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A Class of Linear Partial Neutral Functional Differential Equations with Nondense Domain

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We consider a system of linear partial neutral functional differential equations with nondense domain. A natural generalized notion of solutions is provided by the integral solutions. We derive a variation-of-constants formula which allows us to transform the integral solutions of the neutral equation to solutions of an abstract Volterra integral equation. The basic existence and uniqueness results are given and the solutions are shown to generate an integrated semigroup. © 1998 Academic Press

1. INTRODUCTION

The theory of ordinary NFDE was initiated in Bellman and Cooke [10], Cruz and Hale [13], Hale [19, 20], Hale and Meyer [26], and Henry [27]. They developed the basic theory of existence and uniqueness, as well as properties of the solution operator and stability.

Later, in [15], Datko established the existence of solutions for a class of linear neutral functional differential–difference equations

$$\begin{cases} \dfrac{d}{dt}\left(x(t)-\displaystyle\sum_{i=1}^{n} B_i x(t-h_i)\right)=A_0 x(t)+\displaystyle\sum_{i=1}^{n} C_i x(t-h_i), & t\geqslant 0,\\ x_0=\varphi\in\mathscr{C}_X, \end{cases} \tag{1}$$

defined on a general Banach space X, where $0<h_1<h_2<\cdots<h_n=h$ are given real numbers and $\mathscr{C}_X:=\mathscr{C}([-h,0],X)$ denotes the space of continuous functions from $[-h,0]$ to X with the uniform convergence topology. Let

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$\mathscr{L}(X)$ be the space of bounded linear operators from X into X. Datko assumed that A_0 is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T_0(t))_{t\geqslant 0}$ in X or, equivalently

(i) $\overline{D(A_0)} = X$,

(ii) there exist $M_0, \omega_0 \in \mathbb{R}$ such that if $\lambda > \omega_0 (\lambda I - A_0)^{-1} \in \mathscr{L}(X)$ and

$$\|(\lambda - \omega_0)^m (\lambda I - A_0)^{-m}\|_{\mathscr{L}(X)} \leqslant M_0, \qquad \forall m \in \mathbf{N}.$$

He also assumed that

$$Range(B_i) \subseteq D(A_0) \qquad \text{and} \qquad A_0 B_i \in \mathscr{L}(X), \qquad \text{for each} \quad i = 1, 2, .., n.$$

In this case, the classical semigroup theory ensures the well posedness of Problem (1). Datko proved his results by using the following variation-of-constants formula

$$x(t) = \begin{cases} \sum_{i=1}^{n} B_i x(t-h_i) + T_0(t)\left(\varphi(0) - \sum_{i=1}^{n} B_i \varphi(-h_i)\right) & \\ \qquad + \int_0^t T_0(t-s) \sum_{i=1}^{n} (C_i + A_0 B_i)\, x(s-h_i)\, ds, & \text{if} \quad t \geqslant 0, \\ \varphi(t), & \text{if} \quad t \in [-h, 0], \end{cases} \tag{2}$$

for every $\varphi \in \mathscr{C}_X$. He showed that the solutions of Eq. (2) generate a C_0-semigroup.

More recently, in their study of a ring array of identical resistively coupled transmission lines, Wu and Xia [42, 43] showed that the corresponding system of hyperbolic equations is equivalent to a partial neutral functional differential–difference equation (PNFDDE) defined on the unit circle S^1. They considered equations of the form

$$\frac{\partial}{\partial t}[x(.,t) - qx(.,t-h)] = K \frac{\partial^2}{\partial \xi^2}[x(.,t) - qx(.,t-h)] + f(x_t), \qquad t \geqslant 0, \tag{3}$$

where $x_t(\xi, \theta) = x(\xi, t+\theta)$, $-h \leqslant \theta \leqslant 0$, $t \geqslant 0$, $\xi \in S^1$, K is a positive constant, and $0 \leqslant q < 1$. The space of initial data was chosen to be $\mathscr{C}([-h, 0], H^1(S^1))$. Motivated by this work, Hale presented in [22, 23], the basic theory of existence and uniqueness and properties of the solution operator, as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits for a more general class of PNFDEs on the unit circle S^1. Let us briefly restate the equations considered by Hale in [22, 23].

Let $X = H^1(S^1)$. If $\varphi \in \mathscr{C}_X$, we write it as $\varphi(\xi, \theta)$, for $\xi \in S^1$ and $\theta \in [-h, 0]$. For any function $\tilde{f} \in \mathscr{C}^{k+1}(\mathscr{C}([-h, 0], \mathbb{R}); \mathbb{R})$, $k \geqslant 1$, we let $f \in \mathscr{C}^{k+1}(\mathscr{C}_X, L^2(S^1))$ be defined by $f(\varphi)(\xi) = \tilde{f}(\varphi(\xi, .))$, $\xi \in S^1$. Let $\tilde{D} \in \mathscr{L}(\mathscr{C}([-h, 0], \mathbb{R}); \mathbb{R})$ be defined by

$$\begin{cases} \tilde{D}\psi = \psi(0) - \tilde{g}(\psi), \\ \tilde{g}(\psi) = \displaystyle\int_{-h}^{0} [d_\theta \eta(\theta)]\, \psi(\theta), \end{cases}$$

where η is of bounded variation and nonatomic at 0; that is, there exists a continuous nondecreasing function δ: $[0, h] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left| \int_{-s}^{0} [d_\theta \eta(\theta)]\, \psi(\theta) \right| \leqslant \delta(s)\, \|\psi\|, \qquad s \in [0, h], \quad \psi \in \mathscr{C}([-h, 0], \mathbb{R}).$$

We define $D \in \mathscr{L}(\mathscr{C}_X, X)$ as

$$D(\varphi)(\xi) = \tilde{D}(\varphi(\xi, .)), \qquad \text{for} \quad \xi \in S^1.$$

Hale considered in [22, 23], PNFDE of the form

$$\frac{\partial}{\partial t} Dx_t = K \frac{\partial^2}{\partial \xi^2} Dx_t + f(x_t), \qquad t \geqslant 0, \tag{4}$$

with $\mathscr{C}_X$ as a space of initial data. He considered the Laplace operator $A_0 = K(\partial^2/\partial\xi^2)$ with domain $H^2(S^1)$, which yields an operator verifying (i) and (ii) in X.

In the applications, it is sometimes convenient to take initial functions with more restrictions. There are many examples in concrete situations where evolution equations are not densely defined. Only hypothesis (ii) holds. One can refer for this to [14] or Section 5 for more details. Nondensity occurs, in many situations, from restrictions made on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or from boundary conditions (e.g., the space $\mathscr{C}^1$ with null value on the boundary is nondense in the space of continuous functions). In particular, if instead of $H^1(S^1)$ in the models discussed by Xia and Wu [42, 43], one considers the space of continuous functions

$$\mathscr{C}(S^1) = \underset{\sim}{\mathscr{C}}([0, 1]) = \{u \in \mathscr{C}([0, 1]),\, u(0) = u(1)\}$$

(we are taking S^1 as the homeomorphic image of the quotient of a segment by the relation identifying its endpoints), the domain of the operator A_0 is

$$\mathscr{C}^2(S^1) = \{u \in \mathscr{C}^2([0, 1]), u(0) = u(1) \text{ and } u'(0) = u'(1)\}$$

and the density property is not satisfied in $\mathscr{C}([0, 1])$. Only hypothesis (ii) holds.

In this paper, we consider a general equation of the type of Eq. (3), that is:

$$\frac{d}{dt}(x(t) - Bx(t-h)) = A_0(x(t) - Bx(t-h)) + Dx(t-h) + L(x_t), \qquad t \geqslant 0. \tag{5}$$

More specifically, in most of the paper, we deal with a simpler, seemingly not equivalent, form of Eq. (5), that is to say

$$\frac{d}{dt}(x(t) - Bx(t-h)) = A_0 x(t) + Cx(t-h) + L(x_t), \qquad t \geqslant 0, \tag{6}$$

where X is a Banach space, $A_0 \colon D(A_0) \subseteq X \to X$ a linear operator, $B, C \in \mathscr{L}(X)$, L a continuous linear functional from $\mathscr{C}_X$ into X and x_t denotes the element defined by $x_t(\theta) = x(t+\theta)$, $-h \leqslant \theta \leqslant 0$. The initial value problem associated with Eq. (5) (resp. Eq. (6)) is the following: Given $\varphi \in \mathscr{C}_X$, to find a continuous function $x \colon [-h, b) \to X$, $b > 0$, such that $t \to x(t) - Bx(t-h)$ is differentiable on $[0, b)$, $x(t) - Bx(t-h) \in D(A_0)$ (resp. $x(t) \in D(A_0)$) for $t \in [0, b)$ and x satisfies Eq. (5) (resp. Eq. (6)) for $t \in [0, b)$.

Being simpler, the treatment of Eq. (6) will hopefully be easier to follow. Equation (5), which is the true generalization of Eq. (3), can be solved along the same line as Eq. (6) and is, indeed, covered by our study. This will become apparent in the sequel. One of the main differences is that in dealing with Eq. (6) we need the following assumption

$$Range(B) \subseteq D(A_0), \tag{7}$$

although this assumption is not required for the treatment of Eq. (5). In fact, Condition (7) permits us to write Eq. (6) as an equation (5), with $D = A_0 B + C$. Thanks to the closed graph theorem, (7) implies that

$$A_0 B \in \mathscr{L}(X).$$

The purpose of this paper is to investigate existence, uniqueness, regularity, and continuous dependence on the initial conditions, for Eq. (6), in the case when the operator A_0 satisfies only hypothesis (ii). These results will then be stated without proof for Eq. (5).

Let us now briefly discuss the use of integrated semigroups. In the case where the operators L and D (resp. C) of Eq. (5) (resp. Eq. (6)) are equal to zero, the problem can still be handled by using the classical semigroup theory because A_0 generates a strongly continuous semigroup in the space $\overline{D(A_0)}$. But, if $L \neq 0$ or $D \neq 0$ (resp. $C \neq 0$), it is necessary to impose additional restrictions. A case which is easily handled is when L and D (resp. C) take their values in $\overline{D(A_0)}$. On the other hand, the integrated semigroups theory allows the range of operators L and D (resp. C) to be in a subset of X. As far as we know, no general theory exists in such situations.

We use, in this paper, an approach based on the notion of integrated semigroups, to establish a variation-of-constants formula in the space $\mathscr{C}([-h, 0], X)$, which is the main tool in the development of the local stability and bifurcation theory of equilibrium solutions of nonlinear NFDEs. Let us briefly indicate how it works in Hopf bifurcation theory. In this case, the first step consists naturally in translating the notion of a periodic solution of NFDE into a fixed point problem. As in the case of ordinary delay differential equations (see [1, 4]), the theory of integrated semigroups offers an appropriate tool to perform this transformation (see [1, 4] for further details and [41] for another approach in the case of partial FDE).

In this paper, we discuss only equations with one delay. It is essentially a matter of notation to generalize the theory to any finite number of delays.

After providing some background material in Section 2, we proceed to prove our main abstract results in Section 3. These results give natural generalizations of results in [15, 21, 25, 27]. We also establish, in Theorem 30, that the system described by Eq. (20) generates a locally Lipschitz continuous integrated semigroup on $\mathscr{C}([-h, 0], X)$. The infinitesimal generator of this integrated semigroup is characterized in the same theorem. Section 4 is devoted to Eq. (5). As mentioned earlier, there will be no proof in this section since, obviously, the techniques developed in Section 3 apply without any change. Finally, we give several examples of differential operators with nondense domain satisfying the Hille–Yosida estimates (ii) and we give two examples of PNFDDEs. The first one is associated with the differentiation operator in a one-dimensional compact interval and the second one is associated with the Laplace operator with homogeneous Dirichlet boundary conditions in spaces of continuous functions.

2. INTEGRATED SEMIGROUPS AND DIFFERENTIAL OPERATORS WITH NONDENSE DOMAIN

The purpose of this section is to collect some background materials required throughout this paper. These materials include integrated semigroups theory and differential operators with nondense domain. We will only state results and leave the details to the references.

The following definitions are due to Arendt.

DEFINITION 1. [7] Let X be a Banach space.

A family $(S(t))_{t\geqslant 0}\subset \mathscr{L}(X)$ is called an integrated semigroup if the following conditions are satisfied:

(i) $S(0)=0$;

(ii) for any $x\in X$, $S(t)x$ is a continuous function of $t\geqslant 0$ with values in X;

(iii) for any $t, s\geqslant 0$ $S(s)\, S(t)=\int_0^s (S(t+\tau)-S(\tau))\, d\tau$.

DEFINITION 2. [7] An integrated semigroup $(S(t))_{t\geqslant 0}$ is called exponentially bounded, if there exist constants $M\geqslant 0$ and $\omega\in\mathbb{R}$ such that

$$\|S(t)\|\leqslant Me^{\omega t} \qquad \text{for} \quad t\geqslant 0.$$

Moreover $(S(t))_{t\geqslant 0}$ is called nondegenerate if $S(t)x=0$, for all $t\geqslant 0$, implies that $x=0$.

If $(S(t))_{t\geqslant 0}$ is an integrated semigroup, exponentially bounded, then the Laplace transform $R(\lambda):=\lambda\int_0^{+\infty} e^{-\lambda t}S(t)\, dt$ exists for all λ with $\mathscr{R}e(\lambda)>\omega$. $R(\lambda)$ is injective if and only if $(S(t))_{t\geqslant 0}$ is nondegenerate. $R(\lambda)$ satisfies the following expression

$$R(\lambda)-R(\mu)=(\mu-\lambda)\, R(\lambda)\, R(\mu)$$

and in the case when $(S(t))_{t\geqslant 0}$ is nondegenerate, there exists a unique operator A satisfying $(\omega, +\infty)\subset\rho(A)$ (the resolvent set of A) such that

$$R(\lambda)=(\lambda I-A)^{-1}, \qquad \text{for all} \quad \mathscr{R}e(\lambda)>\omega.$$

This operator A is called the generator of $(S(t))_{t\geqslant 0}$.

We have the following general definition.

DEFINITION 3. [7] An operator A is called a generator of an integrated semigroup, if there exists $\omega\in\mathbb{R}$ such that $(\omega, +\infty)\subset\rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geqslant 0}$ of linear bounded operators such that $S(0)=0$ and $(\lambda I-A)^{-1}=\lambda\int_0^{+\infty} e^{-\lambda t}S(t)\, dt$ for all $\lambda>\omega$.

Remark 1. If an operator A is the generator of an integrated semigroup $(S(t))_{t \geqslant 0}$, then $\forall \lambda \in \mathbb{R}$, $A - \lambda I$ is the generator of the integrated semigroup $(S_\lambda(t))_{t \geqslant 0}$ given by

$$S_\lambda(t) = e^{-\lambda t} S(t) + \lambda \int_0^t e^{-\lambda s} S(s)\, ds.$$

Similar results as for semigroups can be obtained for integrated semigroups.

PROPOSITION 4. [7] *Let A be the generator of an integrated semigroup $(S(t))_{t \geqslant 0}$. Then for all $x \in X$ and $t \geqslant 0$,*

$$\int_0^t S(s)\, x\, ds \in D(A) \qquad \text{and} \qquad S(t)x = A\left(\int_0^t S(s)\, x\, ds\right) + tx.$$

Moreover, for all $x \in D(A)$, $t \geqslant 0$

$$S(t)x \in D(A), \qquad AS(t)x = S(t)\, Ax$$

and

$$S(t)x = \int_0^t S(s)\, Ax\, ds + tx.$$

COROLLARY 5. [7] *Let A be the generator of an integrated semigroup $(S(t))_{t \geqslant 0}$. Then for all $x \in X$ and $t \geqslant 0$ one has $S(t)x \in \overline{D(A)}$.*

Moreover, let $x \in X$. Then $S(\cdot)x$ is right-sided differentiable in $t \geqslant 0$ if and only if $S(t)x \in D(A)$. In that case

$$S'(t)x = AS(t)x + x.$$

One other result is needed for this work.

PROPOSITION 6. [28] *Let $A\colon D(A) \subseteq X \to X$ be a linear operator and $(S(t))_{t \geqslant 0} \subset \mathscr{L}(X)$ an exponentially bounded family. The following assertions are equivalent*

(i) $\int_0^t S(s)x\, ds \in D(A)$ *and* $S(t)x = A(\int_0^t S(s)x\, ds) + tx$, $(t \geqslant 0,\ x \in X)$,

(ii) $(S(t))_{t \geqslant 0}$ *is an integrated semigroup on X generated by A.*

An important special case is when the integrated semigroup is locally Lipschitz continuous (with respect to time).

Definition 7. [33] An integrated semigroup $(S(t))_{t\geqslant 0}$ is called locally Lipschitz continuous, if for all $\tau>0$ there exists a constant $k(\tau)>0$ such that

$$\|S(t)-S(s)\| \leqslant k(\tau)\,|t-s|, \qquad \text{for all} \quad s\in[0,\tau].$$

In this case, we know from [33], that $(S(t))_{t\geqslant 0}$ is exponentially bounded.

Definition 8. [33] We say that a linear operator A satisfies the Hille–Yosida condition (HY) if there exist $M\geqslant 0$ and $\omega\in\mathbb{R}$ such that $(\omega, +\infty)\subset\rho(A)$ and

$$\sup\{(\lambda-\omega)^n\,\|(\lambda I-A)^{-n}\|, n\in\mathbf{N}, \lambda>\omega\}\leqslant M. \tag{HY}$$

The following theorem shows that the Hille–Yosida condition characterizes generators of locally Lipschitz continuous integrated semigroups.

Theorem 9. [33] *The following assertions are equivalent.*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup,*

(ii) *A satisfies the condition (HY).*

In the sequel, we give some results for the existence of solutions of the following Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t)=Au(t)+f(t), & t\geqslant 0,\\ u(0)=x\in X, \end{cases} \tag{8}$$

where A satisfies the condition (HY), without being densely defined.

By a solution of Eq. (8) on $[0,T]$ where $T>0$, we understand a function $u\in\mathscr{C}^1([0,T],X)$ satisfying $u(t)\in D(A)$ for $t\in[0,T]$, such that the two relations in (8) hold.

The following result is due to Da Prato and Sinestrari.

Theorem 10. [14] *Let $A\colon D(A)\subseteq X\to X$ be a linear operator, $f\colon[0,T]\to X$, $x\in D(A)$ such that*

(a) *A satisfies the condition (HY),*

(b) *$f(t)=f(0)+\int_0^t g(s)\,ds$ for some Bochner-integrable function g,*

(c) *$Ax+f(0)\in\overline{D(A)}$.*

Then there exists a unique solution u of Eq. (8) *on the interval* $[0, T]$, *and for each* $t \in [0, T]$

$$|u(t)| \leqslant M e^{\omega t} \left(|x| + \int_0^t e^{-\omega s} \, |f(s)| \, ds \right).$$

In the case where x is not sufficiently regular (that is, x is just in $\overline{D(A)}$) there may not exist a strong solution $u(t) \in X$ but, following the work of Da Prato and Sinestrari [14], Eq. (8) may still have an integral solution. This motives the following definition.

DEFINITION 11. [14] Given $f \in L^1_{loc}(0, +\infty; X)$ and $x \in X$, we say that $u: [0, +\infty) \to X$ is an integral solution of Eq. (8) if the following assertions are true

(i) $u \in \mathscr{C}([0, +\infty); X)$,

(ii) $\int_0^t u(s) \, ds \in D(A)$, for $t \geqslant 0$,

(iii) $u(t) = x + A(\int_0^t u(s) \, ds) + \int_0^t f(s) \, ds$, for $t \geqslant 0$.

From this definition, we deduce that for an integral solution u, we have $u(t) \in \overline{D(A)}$, for all $t > 0$, because $u(t) = \lim_{h \to 0} (1/h) \int_t^{t+h} u(s) \, ds$ and $\int_t^{t+h} u(s) \, ds \in D(A)$. In particular, $x \in \overline{D(A)}$ is a necessary condition for the existence of an integral solution of Eq. (8). This is suggestive to solve Eq. (8) by the variation-of-constants formula

$$u(t) = S'(t) x + \frac{d}{dt} \left(\int_0^t S(t-s) \, f(s) \, ds \right) \qquad \text{for} \quad t \geqslant 0, \tag{9}$$

where $S(t)$ is the integrated semigroup generated by A.

THEOREM 12. [11] *Suppose that A satisfies the condition* (HY), $x \in \overline{D(A)}$ *and* $f: [0, +\infty) \to X$ *is a continuous function. Then the problem* (8) *has a unique integral solution which is given by* (9).

Furthermore, the function u satisfies the inequality

$$|u(t)| \leqslant M e^{\omega t} \left(|x| + \int_0^t e^{-\omega s} \, |f(s)| \, ds \right) \qquad \textit{for} \quad t \geqslant 0.$$

Note that Theorem 12 also says that $\int_0^t S(t-s) \, f(s) \, ds$ is differentiable with respect to t. We have the following general result.

PROPOSITION 13. [3] *Let $(S(t))_{t\geqslant 0}$ be a locally Lipschitz continuous integrated semigroup on a Banach space X and $G\colon [0, T] \to X (T>0)$, a Bochner-integrable function. Then, the function $K\colon [0, T] \to X$ defined by*

$$K(t) = \int_0^t S(t-s)G(s)\, ds$$

is continuously differentiable on $[0, T]$ and satisfies

$$\left|\frac{dK}{dt}(t)\right|_X \leqslant 2k \int_0^t |G(s)|_X\, ds, \qquad \text{for} \quad t\in [0, T],$$

where $k := k(T)$ is the Lipschitz constant of $S(\cdot)$ on $[0, T]$.

3. MAIN RESULTS

We consider a class of linear neutral differential equations, on a Banach space X, of the form

$$\begin{cases} \dfrac{d}{dt}(x(t) - Bx(t-h)) = A_0 x(t) + Cx(t-h) + L(x_t), & t \geqslant 0, \\ x_0 = \varphi \in \mathscr{C}_X, \end{cases} \tag{6}$$

where $A_0\colon D(A_0) \subseteq X \to X$ is a linear operator, $B, C \in \mathscr{L}(X)$, L is a continuous linear functional from $\mathscr{C}_X := \mathscr{C}([-h, 0], X)$ into X and $\varphi \in \mathscr{C}_X$. Equation (6) can be considered as a generalization of the retarded equations ($B=0$) as well as a generalization of difference equation ($A_0 = C = 0$ and $L=0$). We assume that A_0 satisfies (HY) on X, i.e.,

$$\text{there exist } M_0 \geqslant 0 \text{ and } \omega_0 \in \mathbb{R} \text{ such that } (\omega_0, +\infty) \subset \rho(A_0) \text{ and}$$
$$\sup\{(\lambda - \omega_0)^n \, \|(\lambda I - A_0)^{-n}\|, n \in \mathbf{N}, \lambda > \omega_0\} \leqslant M_0. \quad (HY)$$

We assume also that

$$Range(B) \subseteq D(A_0).$$

The closed graph theorem implies that $A_0 B \in \mathscr{L}(X)$.

Theorem 9 shows that A_0 is the generator of a locally Lipschitz continuous integrated semigroup $(S_0(t))_{t\geqslant 0}$ on X, (and $\|S_0(t)\| \leqslant M_0 e^{\omega_0 t}$, for $t \geqslant 0$).

Consider first the linear autonomous neutral differential-difference equation on X

$$\begin{cases} \dfrac{d}{dt}(x(t)-Bx(t-h)) = A_0 x(t) + Cx(t-h), \\ x(t)=\varphi(t), \qquad \text{if} \quad t \geqslant 0, \\ x(t)=\varphi(t), \qquad \text{if} \quad t \in [-h, 0] \qquad \text{and} \qquad \varphi \in \mathscr{C}_X. \end{cases} \tag{10}$$

In the case where C is equal to zero, the problem can still be handled by using the classical semigroup theory because A_0 generates a strongly continuous semigroup in the space $\overline{D(A_0)}$. In the case where $C \neq 0$, it is a little more difficult to define the concept of a solution and the appropriate space of initial data.

In conjunction with the system (10) we consider an integrated form given by the equations

$$y(t) = \begin{cases} By(t-h) + \dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)\, Dy(s-h)\, ds\right) \\ \qquad + S_0(t)\, K(\psi') - B\psi(-h), \qquad \text{if} \quad t \geqslant 0, \\ \psi(t), \qquad \qquad \text{if} \quad t \in [-h, 0], \end{cases} \tag{11}$$

where

$$\psi \in \mathscr{C}_0^1 := \{\psi \in \mathscr{C}^1([-h, 0], X), \psi(0) = 0\},$$

$$D = A_0 B + C$$

and

$$K\colon \mathscr{C}_X \to X,$$

$$\psi \to K(\psi) = \psi(0) - B\psi(-h) + D\left(\int_{-h}^0 \psi(s)\, ds\right).$$

We will discuss the relation between the abstract integral equation (11) and the neutral differential–difference equation (10). Roughly speaking we will prove that if a solution $y := y(., \psi)$ of (11) exists for $\psi \in \mathscr{C}_0^1$ and $\psi'(0) \in \overline{D(A_0)}$ then, y is continuously differentiable for $t \geqslant 0$. Moreover, if we suppose that $A_0 y'(t)$ is defined and continuous for all $t \geqslant 0$ then $x := y'$ satisfies (10) for $\varphi = \psi'$. Also if $x := x(., \varphi)$ is a solution of (10) then $y(t) = \int_0^t x(s)\, ds$ is a solution of (11) for $\psi(\theta) = \int_0^\theta \varphi(s)\, ds$, $\theta \in [-h, 0]$.

If we want to relax the smoothness assumptions on $\varphi \in \mathscr{C}_X$, we may consider Eq. (11) with $\psi(\theta) = \int_0^\theta \varphi(s)\, ds$. In this case, we have $\psi \in \mathscr{C}_0^1$.

THEOREM 14. *Given $\psi \in \mathscr{C}_0^1$, the problem* (11) *has a unique solution y which is a continuous mapping from $[-h, \infty) \to X$.*

Proof. Note that Proposition 13 implies that $\int_0^t S_0(t-s)\, Dy(s-h)\, ds$ is differentiable with respect to t.

It is easy to check that Eq. (11) gives, for $\psi \in \mathscr{C}_0^1$, one possible solution over the interval $[0, h]$. But then y is defined over $[-h, h]$ and since $S_0(t)$ is strongly continuous y is continuous on $[0, h]$ and hence (11) again has a unique continuous solution over $[h, 2h]$. Proceeding inductively we can extend the solution uniquely and continuously to $[0, \infty)$. ∎

Under a smoothing property of φ, we obtain the following result.

THEOREM 15. *If $\varphi \in \mathscr{C}_X$ and $\varphi(0) \in \overline{D(A_0)}$, then the solution $y := y(., \psi)$ of* (11), *with $\psi(\theta) = \int_0^\theta \varphi(s)\, ds$, is continuously differentiable for all $t \geqslant 0$ and y' satisfies the equation*

$$x(t) = \begin{cases} Bx(t-h) + \dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)\, Dx(s-h)\, ds\right) & \\ \qquad + S_0'(t)(\varphi(0) - B\varphi(-h)), & \text{if } t \geqslant 0, \\ \varphi(t), & \text{if } t \in [-h, 0]. \end{cases} \tag{12}$$

Proof. Suppose that $\varphi \in \mathscr{C}_X$ and $\varphi(0) \in \overline{D(A_0)}$.

Then $t \to S_0(t)(\varphi(0) - B\varphi(-h))$ is differentiable for $t \geqslant 0$. Following the proof of Theorem 14, one shows that there exists a unique solution $x := x(., \varphi)$ of (12), which is a continuous mapping from $[-h, \infty) \to X$. Let $w\colon [-h, \infty) \to X$ be the function defined by

$$w(t) = \int_0^t x(s)\, ds.$$

We will show that

$$w = y \qquad \text{on } [-h, \infty).$$

If $t \in [-h, 0]$

$$w(t) = \int_0^t \varphi(s)\, ds = \psi(t),$$

if $t \geqslant 0$, by integrating (12), on both sides from 0 to t, we obtain

$$\begin{aligned}
w(t) &= Bw(t-h) + B\left(\int_{-h}^{0} \varphi(s)\,ds\right) + \int_0^t S_0(t-s)\,Dx(s-h)\,ds \\
&\quad + S_0(t)(\varphi(0) - B\varphi(-h)), \\
&= Bw(t-h) + \frac{d}{dt}\left(\int_0^t S_0(s)\,Dw(t-s-h)\,ds\right) - S_0(t)\,D\psi(-h) \\
&\quad + S_0(t)(\varphi(0) - B\varphi(-h)) - B\psi(-h), \\
&= Bw(t-h) + \frac{d}{dt}\left(\int_0^t S_0(t-s)\,Dw(s-h)\,ds\right) \\
&\quad + S_0(t)\,K(\psi') - B\psi(-h).
\end{aligned}$$

This means that w is also a solution of (11) on $[\,0,\infty)$ with initial value ψ. But the uniqueness property of Theorem 14 implies that

$$y = w \qquad \text{on } [0, \infty).$$

Then y is continuously differentiable for all $t \geqslant 0$ and $y' = x$ satisfies Eq. (12). This completes the proof of Theorem 15. ∎

PROPOSITION 16. *Assume that $\varphi \in \mathscr{C}_X$ and $\varphi(0) \in \overline{D(A_0)}$. Then the solution $x := x(.,\varphi)$ of Problem* (12) *is the unique integral solution of Eq.* (10), *i.e.*

(i) $x \in \mathscr{C}([0, \infty); X)$,

(ii) $\int_0^t x(s)\,ds \in D(A_0)$, *for* $t \geqslant 0$,

(iii) $$x(t) - Bx(t-h) = \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t x(s)\,ds\right) + C\left(\int_0^t x(s-h)\,ds\right), \qquad \textit{for} \quad t \geqslant 0.$$

Proof. Let $x := x(., \varphi)$ be the unique solution of Eq. (12). Consider the function f: $[0, \infty) \to X$ defined by

$$f(s) = Dx(s-h)$$

and the Cauchy problem

$$\begin{cases} u'(t) = A_0 u(t) + f(t), & \text{if} \quad t \geqslant 0, \\ u(0) = \varphi(0) - B\varphi(-h), & \text{if} \quad t = 0. \end{cases} \tag{13}$$

We have

$$\varphi(0) \in \overline{D(A_0)} \qquad \text{and} \qquad B\varphi(-h) \in D(A_0).$$

Then

$$u(0) \in \overline{D(A_0)}$$

and f is a continuous function. Using Theorem 12, we deduce that there exists a unique integral solution u of Eq. (13) which is given by

$$u(t) = S_0'(t)(\varphi(0) - B\varphi(-h)) + \frac{d}{dt}\left(\int_0^t S_0(t-s)\, Dx(s-h)\, ds\right).$$

Since x is the unique solution of Eq. (12), then

$$u(t) = x(t) - Bx(t-h), \qquad \text{for} \quad t \geqslant 0.$$

Moreover, we have

$$\int_0^t u(s)\, ds \in D(A_0).$$

Then, it is easy to verify that

$$\int_0^t x(s)\, ds \in D(A_0),$$

because

$$\mathrm{Range}(B) \subseteq D(A_0).$$

Moreover, we have

$$u(t) = \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t u(s)\, ds\right) + D\left(\int_0^t x(s-h)\, ds\right), \qquad \text{for} \quad t \geqslant 0.$$

Hence

$$\begin{aligned} x(t) - Bx(t-h) = \varphi(0) - B\varphi(-h) &+ A_0\left(\int_0^t x(s)\, ds\right) \\ &+ C\left(\int_0^t x(s-h)\, ds\right), \qquad \text{for} \quad t \geqslant 0 \end{aligned}$$

and the proposition is proved. ∎

Consider the linear operator A: $D(A) \subseteq \mathscr{C}_X \to \mathscr{C}_X$ defined by

$$\begin{cases} D(A) = \{\varphi \in \mathscr{C}^1([-h, 0], X);\ \varphi(0) \in D(A_0), \\ \qquad\qquad \varphi'(0) = A_0\varphi(0) + B\varphi'(-h) + C\varphi(-h)\}, \\ A\varphi = \varphi'. \end{cases} \tag{14}$$

Under more smoothing properties of φ, we obtain the solution of Eq. (10).

THEOREM 17. *Assume that $\varphi \in D(A)$ and $\varphi'(0) \in \overline{D(A_0)}$.*
Let $x := x(., \varphi)$ be the solution of the integral equation (12). *Then x is continuously differentiable for all $t \geqslant 0$ and satisfies Eq.* (10).

Proof. Let $x := x(., \varphi)$ be the solution of (39), and v the unique solution of the problem

$$v(t) = \begin{cases} Bv(t-h) + \dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)\, Dv(s-h)\, ds\right) \\ \qquad + S_0'(t)(\varphi'(0) - B\varphi'(-h)), & \text{if} \quad t \geqslant 0, \\ \varphi'(t), & \text{if} \quad t \in [-h, 0]. \end{cases}$$

Let w: $[-h, \infty) \to X$ be the function defined by

$$w(t) = \varphi(0) + \int_0^t v(s)\, ds, \qquad \text{for} \quad t \geqslant -h.$$

We will show that

$$w = x \qquad \text{on} \quad [-h, \infty).$$

We see at once that, for $t \in [-h, 0]$

$$w(t) = \varphi(0) + \int_0^t \varphi'(s)\, ds = \varphi(t).$$

Using the expression satisfied by φ:

$$\varphi'(0) = A_0\varphi(0) + B\varphi'(-h) + C\varphi(-h),$$

we obtain, for $t \geqslant 0$

$$v(t) = Bv(t-h) + \frac{d}{dt}\left(\int_0^t S_0(t-s)\, Dv(s-h)\, ds\right)$$
$$+ S_0'(t)(A_0\varphi(0) + C\varphi(-h)).$$

By integrating v, we obtain

$$
\begin{aligned}
w(t) &= \varphi(0) + B\left(\int_0^t v(s-h)\,ds\right) + \int_0^t S_0(t-s)\,Dv(s-h)\,ds \\
&\quad + S_0(t)(A_0\varphi(0) + C\varphi(-h)), \\
&= B\left(\int_0^{t-h} v(s)\,ds\right) + \frac{d}{dt}\left(\int_0^t S_0(s)\,D\left(\int_0^{t-s-h} v(\sigma)\,d\sigma\right) ds\right) \\
&\quad + S_0(t)\,D\left(\int_{-h}^0 \varphi'(s)\,ds\right) + S_0(t)(A_0\varphi(0) + C\varphi(-h)) \\
&\quad + \varphi(0) + B\left(\int_{-h}^0 \varphi'(s)\,ds\right), \\
&= Bw(t-h) + \frac{d}{dt}\left(\int_0^t S_0(s)\,Dw(s-h)\,ds\right) \\
&\quad + S_0(t)\,D\left(\int_{-h}^0 \varphi'(s)\,ds\right) + S_0(t)(A_0\varphi(0) + C\varphi(-h)) \\
&\quad + \varphi(0) + B\left(\int_{-h}^0 \varphi'(s)\,ds\right) - B\varphi(0) - S_0(t)\,D\varphi(0).
\end{aligned}
$$

We denote by J the operator defined by

$$
\begin{aligned}
(J\varphi)(t) = S_0(t)\,D\left(\int_{-h}^0 \varphi'(s)\,ds\right) + S_0(t)(A_0\varphi(0) + C\varphi(-h)) \\
+ \varphi(0) + B\left(\int_{-h}^0 \varphi'(s)\,ds\right) - B\varphi(0) - S_0(t)\,D\varphi(0).
\end{aligned}
$$

We have to show that

$$(J\varphi)(t) = S_0'(t)(\varphi(0) - B\varphi(-h)).$$

Since $D = A_0 B + C$, we obtain

$$
\begin{aligned}
(J\varphi)(t) &= -S_0(t)(A_0 B + C)\,\varphi(-h) + S_0(t)(A_0\varphi(0) + C\varphi(-h)) \\
&\quad + \varphi(0) + B(\varphi(0) - \varphi(-h)) - B\varphi(0) \\
&= S_0(t)\,A_0(\varphi(0) - B\varphi(-h)) + \varphi(0) - B\varphi(-h),
\end{aligned}
$$

with $\varphi \in D(A)$, $\varphi'(0) \in \overline{D(A_0)}$ and $\text{Range}(B) \subseteq A_0$.

Thanks to Proposition 4 and Corollary 5, we deduce that

$$(J\varphi)(t) = S_0'(t)(\varphi(0) - B\varphi(-h)).$$

Furthermore, we have

$$w(t) = Bw(t-h) + \frac{d}{dt}\left(\int_0^t S_0(t-s)\,Dw(s-h)\,ds\right)$$
$$+ S_0'(t)(\varphi(0) - B\varphi(-h)).$$

But the uniqueness property of Theorem 15 implies that

$$x = w \qquad \text{on } [0, \infty).$$

This proves that x is continuously differentiable on $[-h, \infty)$. On the other hand we have, form Proposition 16,

$$x(t) - Bx(t-h) = \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t x(s)\,ds\right) + C\left(\int_0^t x(s-h)\,ds\right).$$

Then, we have the existence of the limit

$$\lim_{\sigma\to 0} A_0\left(\frac{1}{\sigma}\int_t^{t+\sigma} x(s)\,ds\right) = \frac{d}{dt}(x(t) - Bx(t-h)) + Cx(t-h).$$

Note that the operator A_0 is closed. Then we have $x(t) \in D(A_0)$ and

$$\frac{d}{dt}(x(t) - Bx(t-h)) = A_0 x(t) + Cx(t-h).$$

The proof of the theorem is complete. ∎

Our next objective is to show that the operator A satisfies the condition (HY) on $\mathscr{C}_X$.

It is easy to prove the following Lemma.

LEMMA 18. *Given $\psi \in \mathscr{C}_0^1$, the solution y of Problem* (11) *given in Theorem* 14 *satisfies, for $t \geqslant 0$*

$$y(t) = B^{p(t)+1}\psi(t-(p(t)+1)h) + \sum_{i=0}^{p(t)} B^i S_0(t-ih)\,K(\psi')$$
$$+ \sum_{i=0}^{p(t)} B^i \frac{d}{dt}\left(\int_0^{t-ih} S_0(t-ih-s)\,Dy(s-h)\,ds\right) - \sum_{i=1}^{p(t)+1} B^i\psi(-h), \tag{15}$$

(*where $p(t) = [t/h]$ and $[\cdot]$ is the whole part function*).

Note that Proposition 13 implies that $\int_0^{t-ih} S_0(t-ih-s)\,Dy(s-h)\,ds$ is differentiable with respect to t.

PROPOSITION 19. *There exist $a \geqslant 0$ and $b \in \mathbb{R}$ such that*

$$|y(t)| \leqslant a e^{bt} \|\psi'\|, \qquad \text{for} \quad t \geqslant 0 \qquad \text{and} \qquad \psi \in \mathscr{C}_0^1.$$

Proof. It follows immediately from Lemma 18 that, for $t \geqslant 0$

$$|y(t)| \leqslant h \|B\|^{p(t)+1} \|\psi'\| + \sum_{i=0}^{p(t)} (\|B\|^i e^{-i\omega_0 h}) e^{\omega_0 t} M_0 \|K\| \|\psi'\|$$

$$+ \sum_{i=0}^{p(t)} (\|B\|^i e^{-i\omega_0 h}) M_0 \|D\| \int_0^t e^{\omega_0 (t-s)} |y(s-h)| \, ds$$

$$+ h \left(\sum_{i=1}^{p(t)+1} \|B\|^i \right) \|\psi'\|.$$

We know that

$$\|B\|^{p(t)+1} \leqslant \sum_{i=1}^{p(t)+1} \|B\|^i, \qquad \text{for all} \quad t \geqslant 0.$$

Moreover, we have

$$\sum_{i=1}^{p(t)+1} \|B\|^i = \begin{cases} \dfrac{\|B\|^{p(t)+1} - 1}{\|B\| - 1}, & \text{if} \quad \|B\| \neq 1, \\ p(t) + 1, & \text{if} \quad \|B\| = 1. \end{cases}$$

Then, there exist $M_1 \geqslant 0$ and $\omega_1 \in \mathbb{R}$ such that

$$\|B\|^{p(t)+1} \leqslant \sum_{i=1}^{p(t)+1} \|B\|^i \leqslant M_1 e^{\omega_1 t}, \qquad \text{for all} \quad t \geqslant 0.$$

Therefore

$$|y(t)| \leqslant 2h M_1 e^{\omega_2 t} \|\psi'\| + \alpha M_0 \|K\| e^{\omega_2 t} \|\psi'\|$$

$$+ \alpha M_0 \|D\| \int_{-h}^{t-h} e^{\omega_2 (t-s-h)} |y(s)| \, ds,$$

where

$$\alpha := \sum_{i=0}^{\infty} (\|B\|^i e^{-i\omega_0 h}) < \infty$$

and

$$\omega_2 = \max(\omega_0, \omega_1, 0).$$

Since

$$y(t) = \psi(t), \qquad \text{for} \quad t \in [-h, 0],$$

we obtain

$$e^{\omega_2 t} |y(t)| \leqslant (2M_1 + h)h \, \|\psi'\| + \alpha M_0 \, \|K\| \, \|\psi'\|$$
$$+ \alpha M_0 \, \|D\| \, e^{-\omega_2 h} \int_0^t e^{-\omega_2 s} |y(s)| \, ds.$$

By Gronwall's inequality, there exist $a \geqslant 0$ and $b \in \mathbb{R}$ such that

$$|y(t)| \leqslant a e^{bt} \, \|\psi'\|$$

and the proposition is proved. ∎

Next, we investigate the abstract properties of the solution of Eq. (11).

Definition 20. If $\varphi \in \mathscr{C}_X$ is given and $y(t) := y(t, \psi)$ is the solution of (11) for $t \geqslant 0$ and $\psi(\theta) = \int_0^\theta \varphi(s) \, ds$ for $\theta \in [-h, 0]$. Define the mapping

$$S\colon [0, \infty) \to \mathscr{L}(\mathscr{C}_X)$$

by the relation

$$S(t)\varphi = y_t(., \psi) - \psi, \tag{16}$$

for $t \geqslant 0$.

Proposition 21. *The family of operators* $(S(t))_{t \geqslant 0}$ *is an integrated semigroup on* $\mathscr{C}_{\mathscr{X}}$ *generated by the operator* A *defined by the relations* (14).

Proof. From Proposition 6, it suffices to show that:

(a) $(S(t))_{t \geqslant 0}$ is exponentially bounded,

(b) $\int_0^t S(s)\varphi \, ds \in D(A)$ and $S(t)\varphi = A(\int_0^t S(s)\varphi \, ds) + t\varphi$, for $t \geqslant 0$ and $\varphi \in \mathscr{C}_X$.

Let $\varphi \in \mathscr{C}_X$ and $y := y(., \psi)$ be the solution of (11), for $\psi = \int_0^{\cdot} \varphi(s)\, ds$. We have

$$(S(t)\varphi)(\theta) = \begin{cases} y(t+\theta) + \int_\theta^0 \varphi(s)\, ds, & t+\theta \geqslant 0, \\ \int_\theta^{t+\theta} \varphi(s)\, ds, & t+\theta < 0. \end{cases} \tag{17}$$

If $t+\theta \leqslant 0$, we obtain

$$|(S(t)\varphi)(\theta)| \leqslant t\, \|\varphi\| \leqslant e^t\, \|\varphi\|.$$

If $t+\theta \geqslant 0$, (17) yields

$$|(S(t)\varphi)(\theta)| \leqslant h\, \|\varphi\| + |y(t+\theta)|.$$

Applying Proposition 19, we obtain

$$|(S(t)\varphi)(\theta)| \leqslant h(1+a)\, e^{bt}\, \|\varphi\|.$$

(b) In this part, we will denote by Ψ the function defined by

$$\Psi(t)(\theta) = \left(\int_0^t S(s)\varphi\, ds\right)(\theta), \qquad \text{for} \quad t \geqslant 0 \qquad \text{and} \qquad \theta \in [-h, 0].$$

We have to show that

$$\Psi(t) \in D(A), \qquad \text{for} \quad t \geqslant 0.$$

It is easy to prove that

$$\Psi(t)(\theta) = \int_\theta^{\theta+t} y(s)\, ds + t \int_\theta^0 \varphi(s)\, ds,$$

$$\theta \to \Psi(t)(\theta) \in \mathscr{C}^1([-h, 0], X), \qquad \text{for each} \quad t \geqslant 0$$

and

$$\frac{d}{d\theta}(\Psi(t)(\theta)) = y(t+\theta) + \int_\theta^0 \varphi(s)\, ds - t\varphi(\theta).$$

We have now

$$\Psi(t)(0) \in D(A_0), \qquad \text{for each} \quad t \geqslant 0.$$

From (17), we deduce that

$$\begin{aligned}\Psi(t)(0) &= \int_0^t y(s)\,ds\\ &= B\left(\int_{-h}^{t-h} y(s)\,ds\right) + \int_0^t S_0(t-s)\,Dy(s-h)\,ds\\ &\quad + \int_0^t S_0(s)\,K(\varphi)\,ds + tB\left(\int_{-h}^0 \varphi(s)\,ds\right).\end{aligned}$$

According to the above assumptions and Proposition 4, we have for $t \geqslant 0$,

$$\int_0^t S_0(t-s)\,Dy(s-h)\,ds \in D(A_0),$$

$$\int_0^t S_0(s)\,K(\varphi)\,ds \in D(A_0)$$

and

$$B\left(\int_{-h}^{t-h} y(s)\,ds\right) + tB\left(\int_{-h}^0 \varphi(s)\,ds\right) \in D(A_0).$$

Then

$$\Psi(t)(0) \in D(A_0), \qquad \text{for} \quad t \geqslant 0.$$

We can now introduce the following function

$$H(t) := \frac{d}{d\theta}\Psi(t)(0) - A_0(\Psi(t)(0)) - B\left(\frac{d}{d\theta}\Psi(t)(-h)\right) - C(\Psi(t)(-h)),$$
$$\text{for} \quad t \geqslant 0.$$

We have to show that

$$H(t) = 0, \qquad \text{for} \quad t \geqslant 0.$$

It is immediate that

$$H(t)=y(t)-t\varphi(0)-A_0\left(\int_0^t y(s)\,ds\right)$$
$$-B\left(y(t-h)+\int_{-h}^0 \varphi(s)\,ds-t\varphi(-h)\right)$$
$$-C\left(\int_{-h}^{t-h} y(s)\,ds+t\int_{-h}^0 \varphi(s)\,ds\right).$$

Note that if we consider the function Y defined, for $t\geqslant 0$, by

$$Y(t)=y(t)-B\left(y(t-h)+\int_{-h}^0 \varphi(s)\,ds\right). \tag{18}$$

We have from (11)

$$\begin{cases} Y(t)=\dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)(Dy(s-h)+K(\varphi))\,ds\right), & \text{for} \quad t\geqslant 0,\\ Y(0)=0. \end{cases}$$

By Theorem 12, we obtain, for $t\geqslant 0$

$$Y(t)=A_0\left(\int_0^t Y(s)\,ds\right)+\int_0^t f(s)\,ds,$$

where

$$f(s)=Dy(s-h)+K(\varphi), \qquad \text{for} \quad s\geqslant 0.$$

Therefore

$$Y(t)=A_0\left(\int_0^t Y(s)\,ds\right)+D\left(\int_0^t y(s-h)\,ds\right)+tK(\varphi).$$

We have, for $t\geqslant 0$

$$H(t)=Y(t)-A_0\left(\int_0^t y(s)\,ds-C\left(\int_{-h}^{t-h} y(s)\,ds\right)\right.$$
$$\left.+t\left(B\varphi(-h)-\varphi(0)-C\left(\int_{-h}^0 \varphi(s)\,ds\right)\right)\right).$$

Then, we obtain

$$H(t)=A_0\left(\int_0^t Y(s)-y(s)\,ds\right)+D\left(\int_{-h}^{t-h} y(s)\,ds\right)$$
$$+t\left(B\varphi(-h)-\varphi(0)-C\left(\int_{-h}^{0}\varphi(s)\,ds\right)+K(\varphi)\right)$$
$$-C\left(\int_{-h}^{t-h} y(s)\,ds\right).$$

From (18), the expression of D and the expression of $K(\varphi)$, we deduce

$$H(t)=-A_0B\left(\int_0^t\left(y(s-h)+\int_{-h}^{0}\varphi(u)\,du\right)ds\right)+A_0B\left(\int_{-h}^{t-h} y(s)\,ds\right)$$
$$+C\left(\int_{-h}^{t-h} y(s)\,ds\right)-C\left(\int_{-h}^{t-h} y(s)\,ds\right),$$
$$=0.$$

We conclude that

$$\int_0^t S(s)\,\varphi\,ds\in D(A),\qquad \text{for}\quad t\geqslant 0.$$

On the other hand, it is easy to show that

$$\left(S(t)\varphi-A\left(\int_0^t S(s)\,\varphi\,ds\right)-t\varphi\right)(\theta)=0,\qquad \text{for}\quad t\geqslant 0\quad\text{and}\quad \theta\in[-h,0].$$

This finishes the proof of the proposition. ∎

THEOREM 22. *The operator A satisfies the condition (HY) on $\mathscr{C}_X$.*

Proving directly that A satisfies (HY) seems to be more complicated than proving that its integrated semigroup $(S(t))_{t\geqslant 0}$ is locally Lipschitz continuous.

We need the following Lemma.

LEMMA 23. *For all $\alpha>0$ there exists a constant $c(\alpha)\geqslant 0$ such that the solution $y:=y(.,\psi)$ of* (11), *with $\psi=\int_0^{\cdot}\varphi(s)\,ds$, satisfies*

$$|y(t+\tau)-y(t)|\leqslant c(\alpha)\,\tau\,\|\varphi\|,$$

for all t, $t+\tau\in[0,\alpha]$ and $\tau\geqslant 0$.

Proof of Lemma 23. Consider $\alpha > 0$, $t, t+\tau \in [0, \alpha]$, $\tau \geqslant 0$ and $\varphi \in \mathscr{C}_X$. We have

$$\begin{aligned}
& y(t+\tau) - y(t) \\
&\quad = B^{p(t)+1}[y(t-(p(t)+1)h+\tau) - y(t-(p(t)+1)h)] \\
&\qquad + \sum_{i=0}^{p(t)} B^i \frac{d}{dt}\left(\int_{t-ih}^{t+\tau-ih} S_0(t+\tau-ih-s)\, Dy(s-h)\, ds\right) \\
&\qquad + \sum_{i=0}^{p(t)} B^i \frac{d}{dt}\left(\int_0^{t-ih} (S_0(t+\tau-ih-s) - S_0(t-ih-s))\, Dy(s-h)\, ds\right) \\
&\qquad + \sum_{i=0}^{p(t)} B^i (S_0(t+\tau-ih) - S_0(t-ih))\, K(\varphi),
\end{aligned}$$

where $p(t) = [t/h]$.

A trivial verification shows that

$$\begin{aligned}
& |y(t+\tau) - y(t)| \\
&\quad \leqslant M_1(\alpha)\, |y(t+\tau-(p(t)+1)h) - y(t-(p(t)+1)h)| \\
&\qquad + M_2(\alpha) \sum_{i=0}^{p(t)} \int_{t-ih}^{t+\tau-ih} \|S_0(t+\tau-ih-s)\|\, |y(s-h)|\, ds \\
&\qquad + M_3(\alpha) \sum_{i=0}^{p(t)} \int_0^{\alpha} \|S_0(t+\tau-ih-s) - S_0(t-ih-s)\|\, |y(s-h)|\, d \\
&\qquad + M_4(\alpha) \sum_{i=0}^{p(t)} \|S_0(t+\tau-ih) - S_0(t-ih)\|\, \|\varphi\|.
\end{aligned}$$

If we put $\mu = t-(p(t)+1)h$, we remark that $\mu \in [-h, 0]$ and $\mu+\tau \in [-h, \alpha]$.

If $\mu+\tau \in [-h, 0]$, we obtain

$$|y(\mu+\tau) - y(\mu)| = \left|\int_{\mu}^{\mu+\tau} \varphi(s)\, ds\right| \leqslant \tau\, \|\varphi\|.$$

If $\mu+\tau \in [0, \alpha]$, we obtain

$$\begin{aligned}
y(\mu+\tau) - y(\mu) = B(y(\mu+\tau-h) - y(-h)) & \\
+ \frac{d}{dt}\left(\int_0^{\mu+\tau} S_0(\mu+\tau-s)\, Dy(s-h)\, ds\right) & \\
+ S_0(\tau+\mu)\, K(\varphi) + \int_{\mu}^{0} \varphi(s)\, ds, &
\end{aligned}$$

because

$$y(\theta)=\int_0^{\theta}\varphi(s)\,ds, \qquad \text{for} \quad \theta\in[-h,0].$$

Then

$$\begin{aligned}|y(\mu+\tau)-y(\mu)| \leqslant \|B\|\ |y(\mu+\tau-h)-y(-h)| & \\ +k(\alpha)\ \|D\|\ (\mu+\tau)\left(\int_{-h}^{\mu+\tau-h}|y(s)|\,ds\right) & \\ +k(\alpha)\ \|K\|\ (\mu+\tau)\ \|\varphi\|-\mu\ \|\varphi\|, &\end{aligned}$$

because $(S_0(t))_{t\geqslant 0}$ is Lipschitz continuous on $[0,\alpha]$.

So, we deduce that

$$|y(\mu+\tau)-y(\mu)| \leqslant \|B\|\ |y(\mu+\tau-h)-y(-h)|+G(\alpha)\tau\ \|\varphi\|,$$

where

$$G(\alpha)=k(\alpha)\ \|D\|\ C(\alpha)+k(\alpha)\ \|K\|+1,$$

and

$$C(\alpha)=h+\alpha a e^{b\alpha}.$$

Furthermore, we have for $\mu+\tau\in[0,h]$

$$|y(\mu+\tau-h)-y(-h)| \leqslant (\mu+\tau)\ \|\varphi\| \leqslant \tau\ \|\varphi\|.$$

So, we deduce that

$$|y(\mu+\tau)-y(\mu)| \leqslant (\|B\|+G(\alpha))\tau\ \|\varphi\|.$$

For $\mu+\tau\in[h,2h]$,

$$|y(\mu+\tau-h)-y(-h)| \leqslant (\|B\|+G(\alpha))(\mu+\tau)\ \|\varphi\| \leqslant (\|B\|+G(\alpha))\tau\ \|\varphi\|,$$

and we obtain

$$\begin{aligned}|y(\mu+\tau)-y(\mu)| &\leqslant [\,\|B\|(\|B\|+G(\alpha))+G(\alpha)]\,\tau\ \|\varphi\|,\\ &=[\,\|B\|^2+(\|B\|+1)\ G(\alpha)]\,\tau\ \|\varphi\|.\end{aligned}$$

For $\mu+\tau\in[ph,(p+1)h]$, we obtain

$$|y(\mu+\tau)-y(\mu)| \leqslant \left[\ \|B\|^{p+1}+\left(\sum_{i=0}^{p}\|B\|^i\right)G(\alpha)\right]\tau\ \|\varphi\|.$$

It is easy to check that there exists a constant $\beta(\alpha) \geqslant 0$, such that

$$\|B\|^{p+1} + \left(\sum_{i=0}^{p} \|B\|^{i} \right) G(\alpha) \leqslant \beta(\alpha).$$

This proves the lemma. ∎

Proof of Theorem 22. Consider $\alpha > 0$, $t, s \in [0, \alpha]$ and $\varphi \in \mathscr{C}_X$. If $t + \theta \leqslant 0$ and $s + \theta \leqslant 0$, we have

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| = 0.$$

If $t + \theta \geqslant 0$ and $s + \theta \leqslant 0$, we obtain

$$(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta) = y(t+\theta) + \int_{s+\theta}^{0} \varphi(u)\, du.$$

Then Lemma 23 implies

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| \leqslant c(\alpha)(t+\theta)\ \|\varphi\| - (s+\theta)\ \|\varphi\|.$$

Then

$$\|S(t) - S(s)\| \leqslant (c(\alpha) + 1)\ |t - s|.$$

If $t + \theta \geqslant 0$ and $s + \theta \geqslant 0$, we have

$$(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta) = y(t+\theta) - y(s+\theta).$$

It follows immediately from Lemma 23 that there exists a constant $l := l(\alpha) > 0$ such that

$$|(S(t)\varphi)(\theta) - (S(s)\varphi)(\theta)| \leqslant l(\alpha)\ |t - s|\ \|\varphi\|.$$

It may be concluded that $(S(t))_{t \geqslant 0}$ is locally Lipschitz continuous. The proof of Theorem 22 is complete. ∎

Consider now the linear autonomous neutral differential equation on X

$$\begin{cases} \dfrac{d}{dt}(x(t) - Bx(t-h)) = A_0 x(t) + Cx(t-h) + L(x_t), & t \geqslant 0, \\ x_0 = \varphi \in \mathscr{C}_X, \end{cases} \tag{6}$$

We will investigate a variation-of-constants formula associated to Eq. (6). We need to extend the integrated semigroup $(S(t))_{t \geqslant 0}$ to the space

$$\widetilde{\mathscr{C}}_X = \mathscr{C}_X \oplus \langle X_0 \rangle,$$

where $\langle X_0 \rangle = \{X_0 c,\ c \in X$ and $(X_0 c)(\theta) = X_0(\theta) c\}$ and X_0 denotes the function defined by

$$X_0(\theta) = \begin{cases} 0, & \text{if} \quad \theta < 0, \\ Id_X, & \text{if} \quad \theta = 0. \end{cases}$$

To define a fundamental integral solution $Z(t)$, one proceeds in a manner similar to the one used for ordinary NFDE.

Consider first the integral equation

$$z(t) = \begin{cases} Bz(t-h) + \dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)\, Dz(s-h)\, ds\right) + S_0(t) c, \\ \qquad \text{if} \quad t \geqslant 0, \\ 0, \qquad \text{if} \quad t \in [-h, 0], \end{cases} \tag{19}$$

where $c \in X$ is given.

Following the proofs of Theorem 14 and Proposition 19, one shows the following result.

Proposition 24. *Given $c \in X$, the problem* (19) *has a unique solution $z := z(., c)$ which is a continuous mapping from $[-h, \infty) \to X$. Moreover the operator $Z(t)$: $X \to X$ defined by*

$$Z(t) c = z(t, c)$$

satisfies the inequality

$$\|Z(t)\| \leqslant \alpha e^{\beta t}, \qquad \textit{for} \quad t \geqslant 0.$$

Let us consider the family of operators $(\tilde{S}(t))_{t \geqslant 0}$ defined on $\tilde{\mathscr{C}}_X$ by

$$\tilde{S}(t) \varphi = S(t) \varphi, \qquad \text{for} \quad \varphi \in \mathscr{C}_X$$

and

$$(\tilde{S}(t)\, X_0 c)(\theta) = \begin{cases} Z(t+\theta) c, & \text{if} \quad t+\theta \geqslant 0, \\ 0, & \text{if} \quad t+\theta \leqslant 0, \quad \text{for } c \in X. \end{cases}$$

We shall prove that this extension determines a locally Lipschitz continuous integrated semigroup on $\tilde{\mathscr{C}}_X$.

PROPOSITION 25. *$(\widetilde{S}(t))_{t\geqslant 0}$ is a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathscr{C}}_X$ generated by the operator $\widetilde{A}$ defined by*

$$\begin{cases} D(\widetilde{A}) = \{\varphi \in \mathscr{C}^1([-r, 0], X);\ \varphi(0) \in D(A_0)\}, \\ \widetilde{A}\varphi = \varphi' + X_0(A_0\varphi(0) + B\varphi'(-h) - \varphi'(0) + C\varphi(-h)). \end{cases}$$

Proof. Using the same reasoning as in the proof of Proposition 21 and Theorem 22 one can show that $(\widetilde{S}(t))_{t\geqslant 0}$ is a locally Lipschitz continuous integrated semigroup on $\widetilde{\mathscr{C}}_X$ generated by $\widetilde{A}$. We can also use Proposition 26 to prove this result. ∎

For each complex λ, we define the linear operator

$$\Delta(\lambda)\colon D(A_0) \to X$$

by

$$\Delta(\lambda) \stackrel{\text{def}}{=} \lambda(I - e^{-h\lambda}B) - A_0 - e^{-h\lambda}C.$$

We have the following result.

PROPOSITION 26. *There exists $\omega \in \mathbb{R}$ such that, for $\lambda > \omega$, one has*

(i) $D(\widetilde{A}) = D(A) \oplus \langle e^{\lambda\cdot} \rangle$, *where*

$$\langle e^{\lambda\cdot} \rangle = \{e^{\lambda\cdot}c;\ c \in D(A_0),\ (e^{\lambda\cdot}c)(\theta) = e^{\lambda\theta}c\},$$

(ii) $(\omega, +\infty) \subset \rho(\widetilde{A})$ *and*

$$(\lambda I - \widetilde{A})^{-1}(\varphi + X_0 c) = (\lambda I - A)^{-1}\varphi + e^{\lambda\cdot}\Delta(\lambda)^{-1}c,$$

for every $(\varphi, c) \in \mathscr{C}_X \times X$.

Proof. We have, for $\lambda > 0$

$$\Delta(\lambda) = \lambda(I - e^{-h\lambda}B) - A_0 - e^{-h\lambda}C = \lambda\left(I - \frac{1}{\lambda}L_\lambda\right).$$

with

$$L_\lambda = \lambda e^{-h\lambda}B + A_0 + e^{-h\lambda}C.$$

Then

$$\|L_\lambda\| \leqslant (\lambda\|B\| + \|C\|)\, e^{-h\lambda} + \|A_0\|.$$

There exist $a, b > 0$, such that

$$\|L_\lambda\| < a, \qquad \text{for} \quad \lambda > b.$$

So,

$$\left\| \frac{1}{\lambda} L_\lambda \right\| < 1, \qquad \text{for} \quad \lambda > \bar{\omega} = \max(a, b).$$

Hence the operator $\Delta(\lambda)$ is invertible, for $\lambda > \bar{\omega}$.

For the proof of (i), we consider the following operator

$$\begin{aligned} l \colon D(\tilde{A}) &\to X \\ \varphi &\to l(\varphi) = A_0 \varphi(0) + C\varphi(-h) + B\varphi'(-h) - \varphi'(0). \end{aligned}$$

Let $\tilde{\Psi} \in D(\tilde{A})$ and $\lambda > \bar{\omega}$. Setting $\Psi = \tilde{\Psi} - e^{\lambda \cdot} \Delta(\lambda)^{-1} l(\tilde{\Psi})$, we deduce that $\Psi \in Ker(l) = D(A)$, and the decomposition is clearly unique.

(ii) Consider the equation

$$(\lambda I - \tilde{A})(\varphi + e^{\lambda \cdot} c) = \psi + X_0 a,$$

where $(\psi, a) \in \mathscr{C}_X \times X$ is given and we are looking for $(\varphi, c) \in D(A) \times D(A_0)$. This yields

$$(\lambda I - A)\varphi + \lambda e^{\lambda \cdot} c - \lambda e^{\lambda \cdot} c + X_0 \Delta(\lambda) c = \psi + X_0 a.$$

Then, there exists $\omega > 0$, such that

$$\begin{cases} \varphi = (\lambda I - A)^{-1} \psi, \\ c = \Delta(\lambda)^{-1} a, \qquad \text{for} \quad \lambda > \omega. \end{cases}$$

Consequently,

$$\begin{cases} (\omega, +\infty) \subset \rho(\tilde{A}), \\ (\lambda I - \tilde{A})^{-1} (\varphi + X_0 c) = (\lambda I - A)^{-1} \varphi + e^{\lambda \cdot} \Delta(\lambda)^{-1} c. \end{cases}$$

The proof of the proposition is complete. ∎

Corollary 27. *The linear operator $Z(t)$ is the fundamental integral solution; that is*

$$\Delta(\lambda)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} Z(t)\, dt, \qquad \textit{for} \quad \lambda > \omega.$$

Proof. We have, for $c \in X$

$$(\lambda I - \tilde{A})^{-1} (X_0 c) = e^{\lambda \cdot} \Delta(\lambda)^{-1} c.$$

Then

$$e^{\lambda\theta}\Delta(\lambda)^{-1}c=\lambda\int_0^{+\infty}e^{-\lambda s}(\tilde{S}(s)\,X_0c)(\theta)\,ds=\lambda\int_{-\theta}^{+\infty}e^{-\lambda s}Z(s+\theta)\,c\,ds.$$

So,

$$\Delta(\lambda)^{-1}c=\lambda\int_0^{+\infty}e^{-\lambda t}Z(t)\,c\,dt.$$

This completes the proof. ∎

It is easy to prove the following result.

Corollary 28. *If* $c\in\overline{D(A_0)}$ *then the function*

$$t\to z(t)=Z(t)\,c$$

is differentiable for all $t>0$ *and we have*

$$\Delta(\lambda)^{-1}c=\int_0^{+\infty}e^{-\lambda t}Z'(t)\,dt.$$

Hence the name of fundamental integral solution.

The fundamental solution $Z'(t)$ is defined only for $c\in\overline{D(A_0)}$ and is discontinuous at zero. It is expected that this discontinuity will persist at multiples of h. For all these reasons we prefer to use the fundamental integral solution $Z(t)$.

Our next objective is to obtain a representation of the solution of Eq. (6) in terms of the fundamental integral solution $Z(t)$, or equivalently in terms of $\tilde{S}(t)\,X_0$.

Theorem 29. *Given* $\varphi\in\mathscr{C}_X$, *there exists a unique function* $u:=u(.,\varphi)$: $[0,\infty)\to\mathscr{C}_X$ *which solves the following abstract integral equation*

$$u(t)=S(t)\varphi+\frac{d}{dt}\left(\int_0^t\tilde{S}(t-s)\,X_0L(u(s))\,ds\right),\qquad \textit{for}\quad t\geqslant 0. \tag{20}$$

Proof. Note that Proposition 13 implies that $\int_0^t\tilde{S}(t-s)\,X_0L(y(s))\,ds$ is differentiable with respect to t. Let $T>0$ and $(u^n)_{n\in\mathbf{N}}$ be a sequence of continuous functions defined by

$$u^0(t)=S(t)\varphi,\qquad t\in[0,T]$$

$$u^n(t)=S(t)\varphi+\frac{d}{dt}\left(\int_0^t\tilde{S}(t-s)\,X_0L(u^{n-1}(s))\right)ds,\qquad t\in[0,T],\quad n\geqslant 1.$$

By virtue of the continuity of L and $S(\cdot)\varphi$, there exists $\alpha \geqslant 0$ such that $|L(S(s)\varphi| \leqslant \alpha$, for $s \in [0, T]$. Then, using Proposition 13, we obtain

$$|u^1(t) - u^0(t)| \leqslant 2k \int_0^t |L(S(s)\varphi|\, ds,$$

hence

$$|u^1(t) - u^0(t)| \leqslant 2k\alpha t.$$

In general case we have

$$|u^n(t) - u^{n-1}(t)| \leqslant 2k\, \|L\| \int_0^t |u^{n-1}(s) - u^{n-2}(s)|\, ds,$$

so,

$$|u^n(t) - u^{n-1}(t)| \leqslant 2^n k^n\, \|L\|^{n-1}\, \alpha \frac{t^n}{n!}.$$

Consequently, the limit $u := \lim_{n \to \infty} u^n(t)$ exists uniformly on $[0, T]$ and u is continuous on $[0, T]$.

In order to prove that u is a solution of Eq. (20), we introduce the function v defined by

$$v(t) = \left| u(t) - S(t)\varphi - \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(u(s))\, ds \right) \right|.$$

We have

$$\begin{aligned} v(t) &\leqslant |u(t) - u^{n+1}(t)| + \left| u^{n+1}(t) - S(t)\varphi - \frac{d}{dt} \int_0^t \tilde{S}(t-s)\, X_0 L(u(s))\, ds \right|, \\ &\leqslant |u(t) - u^{n+1}(t)| + \left| \frac{d}{dt} \int_0^t \tilde{S}(t-s)\, X_0 L(u(s) - u^n(s))\, ds \right|, \\ &\leqslant |u(t) - u^{n+1}(t)| + 2k\, \|L\| \int_0^t |u(t) - u^n(t)|. \end{aligned}$$

Moreover, we have

$$u(t) - u^n(t) = \sum_{p=0}^{\infty} (u^{p+1}(t) - u^p(t)).$$

This implies that

$$v(t) \leqslant (1+2k\,\|L\|)\,\frac{\alpha}{\|L\|}\sum_{p=n}^{\infty}(2k\,\|L\|)^{p+1}\,\frac{t^{p+1}}{(p+1)!}$$

$$+\frac{\alpha}{\|L\|}\,(2k\,\|L\|)^{n+1}\,\frac{t^{n+1}}{(n+1)!}, \qquad \text{for} \quad n \in \mathbf{N}.$$

Consequently we obtain $v=0$ on $[0, T]$.

To show uniqueness, suppose that $w(t)$ is also a solution of (20). Then

$$|u(t)-w(t)| \leqslant 2k\,\|L\| \int_0^t |u(s)-w(s)|\,ds.$$

By Gronwall's inequality, $w=u$ on $[0, T]$. ∎

THEOREM 30. *The family of operators* $(U(t))_{t \geqslant 0}$ *defined on* $\mathscr{C}_X$ *by*

$$U(t)\,\varphi = u(t, \varphi)$$

is a locally Lipschitz continuous integrated semigroup on $\mathscr{C}_X$ *generated by the operator* P *defined by*

$$\begin{cases} D(P) = \{\varphi \in \mathscr{C}^1([-h, 0], X);\ \varphi(0) \in D(A_0), \\ \qquad\qquad \varphi'(0) = A_0\varphi(0) + C\varphi(-h) + B\varphi'(-h) + L(\varphi)\}, \\ P\varphi = \varphi'. \end{cases}$$

Proof. Consider the operator

$$\tilde{L}\colon \mathscr{C}_X \to \tilde{\mathscr{C}}_X$$

defined by

$$\tilde{L}(\varphi) = X_0 L(\varphi).$$

Using a result of Kellermann [32], one can prove that the operator $\tilde{G}$ defined in $\tilde{\mathscr{C}}_X$ by

$$\begin{cases} D(\tilde{G}) = D(\tilde{A}), \\ \tilde{G} = \tilde{A} + \tilde{L}, \end{cases}$$

is the generator of a locally Lipschitz continuous integrated semigroup on $\tilde{\mathscr{C}}_X$, because $\tilde{A}$ satisfies (HY) and $\tilde{L} \in \mathscr{L}(\overline{D(\tilde{G})}, \tilde{\mathscr{C}}_X)$, with $\overline{D(\tilde{G})} = \mathscr{C}_X$.

Let us introduce the part G of $\tilde{G}$ in $\mathscr{C}_X$, which is defined by:

$$\begin{cases} D(G) = \{\varphi \in D(\tilde{G});\ \tilde{G}\varphi \in \mathscr{C}_X\}, \\ G(\varphi) = \tilde{G}(\varphi). \end{cases}$$

It is easy to see that

$$G = P.$$

Then, P is the generator of a locally Lipschitz continuous integrated semigroup $(V(t))_{t \geqslant 0}$ on $\mathscr{C}_X$.

On the other hand if we consider, for each $\varphi \in \mathscr{C}_X$, the nonhomogeneous Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) = \tilde{A}u(t) + h(t), \qquad \text{for} \quad t \geqslant 0, \\ u(0) = 0, \end{cases} \tag{21}$$

where h: $[0, +\infty[\rightarrow \tilde{\mathscr{C}}_X$ is given by

$$h(t) = \varphi + \tilde{L}(V(t)\varphi).$$

By Theorem 12, the nonhomogeneous Cauchy problem (21) has a unique integral solution u given by

$$\begin{aligned} u(t) &= \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, h(s)\, ds\right), \\ &= \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\varphi\, ds\right) + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(V(s)\varphi)\, ds\right). \end{aligned}$$

Then

$$u(t) = S(t)\varphi + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(V(s)\varphi)\, ds\right).$$

On the other hand, we have

$$V(t)\varphi = P\left(\int_0^t V(s)\varphi\, ds\right) + t\varphi.$$

This implies that

$$\frac{d}{d\theta}\left(\int_0^t V(s)\varphi\, ds\right) = V(t)\,\varphi - t\varphi.$$

For $\psi \in D(P)$, we have

$$P\psi = \tilde{A}\psi + X_0 L(\psi).$$

Then, we obtain

$$V(t)\varphi = \tilde{A}\left(\int_0^t V(s)\varphi\, ds\right) + X_0 L\left(\int_0^t V(s)\varphi\, ds\right) + t\varphi,$$

so,

$$V(t)\varphi = \tilde{A}\left(\int_0^t V(s)\varphi\, ds\right) + \int_0^t h(s)\, ds.$$

Hence, the function $t \to V(t)\varphi$ is an integral solution of (21). By uniqueness, we conclude that $V(t)\varphi = u(t)$, for all $t \geqslant 0$. By Theorem 29, we have $U(t) = V(t)$ on $\mathscr{C}_X$. Thus the proof of Theorem 30. ▮

Using the same reasoning as in Theorem 29, one can prove the following Proposition.

PROPOSITION 31. *For given $\varphi \in \mathscr{C}_X$ such that $\varphi(0) \in \overline{D(A_0)}$, the solution u of the problem* (20) *is continuously differentiable for all $t \geqslant 0$ and u' satisfies the equation*

$$v(t) = S'(t)\varphi + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(v(s))\, ds\right), \qquad \text{for} \quad t \geqslant 0. \tag{22}$$

COROLLARY 32. *Under the same assumptions as in Proposition* 31, *the solution v of the integral equation* (22) *satisfies, for $t \geqslant 0$ and $\theta \in [-h, 0]$ the translation property*

$$v(t)(\theta) = \begin{cases} v(t+\theta)(0), & \text{if} \quad t+\theta \geqslant 0, \\ \varphi(t+\theta), & \text{if} \quad t+\theta \leqslant 0. \end{cases}$$

Moreover, if we consider the function x: $[-h, \infty) \to X$ defined by

$$x(t) = \begin{cases} v(t)(0), & \text{if} \quad t \geqslant 0, \\ \varphi(t), & \text{if} \quad t \leqslant 0. \end{cases}$$

Then, x is the unique integral solution of Eq. (6), *i.e.*,

(i) $x \in \mathscr{C}([0, \infty); X)$,

(ii) $\int_0^t x(s)\, ds \in D(A_0)$, *for* $t \in [0, \infty)$,

(iii) $x(t) - Bx(t-h)$

$$= \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t x(s)\,ds\right)$$

$$+ C\left(\int_0^t x(s-h)\,ds\right) + L\left(\int_0^t x_s\,ds\right), \qquad \text{for} \quad t \in [0, \infty).$$

Furthermore, there exist $\gamma \geqslant 0$ *and* $\mu \in \mathbb{R}$ *such that*

$$\|x_t\| \leqslant \gamma e^{\mu t}\,\|\varphi\|, \qquad \text{for} \quad t \geqslant 0.$$

Proof. We have, for $t + \theta \geqslant 0$

$$v(t)(\theta) = y(t+\theta) + \frac{d}{dt}\left(\int_0^{t+\theta} z(t+\theta-s)(L(v(s)))\,ds\right)$$

$$= v(t+\theta)(0)$$

and for $t + \theta \leqslant 0$, we have

$$\left(\int_0^t \tilde{S}(t-s)\,X_0 L(v(s))\,ds\right)(\theta) = 0$$

and

$$v(t)(\theta) = \varphi(t+\theta).$$

If we consider the function $f\colon [0, \infty) \to \mathscr{C}_X$, defined by

$$f(s) = X_0 L(v(s)),$$

We can use Theorem 12 to prove that v is the unique integral solution of the equation

$$\begin{cases} v'(t) = \tilde{A}v(t) + f(t), & t \geqslant 0, \\ v(0) = \varphi. \end{cases}$$

If we take $\theta = 0$, we obtain (i), (ii), and (iii).

If we consider now, the function $x\colon [-h, \infty) \to X$ defined by

$$x(t) = \begin{cases} v(t)(0), & \text{if} \quad t \geqslant 0, \\ \varphi(t), & \text{if} \quad t \leqslant 0. \end{cases}$$

we obtain

$$v(t) = x_t.$$

Then

$$x_t = S'(t)\varphi + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0(L(x_s))\, ds\right).$$

This yields

$$\|x_t\| \leqslant M\left(\|\varphi\| + \|L\| \int_0^t e^{-\omega s}\, \|x_s\|\, ds\right) e^{\omega t},$$

By Gronwall's inequality, we obtain

$$\|x_t\| \leqslant \gamma e^{\mu t}\, \|\varphi\|.$$

The proof is finished. ∎

THEOREM 33. *Given* $\varphi \in \mathscr{C}_X$ *such that*

$$\varphi(0) \in D(A_0),\ \varphi' \in \mathscr{C}_X,\ \varphi'(0) \in \overline{D(A_0)} \qquad \text{and}$$
$$\varphi'(0) = A_0\varphi(0) + B\varphi'(-h) + C\varphi(-h) + L(\varphi),$$

let u: $[0, \infty) \to \mathscr{C}_X$ *be the solution of the abstract integral equation* (22) *such that* $u(0) = \varphi$. *Then,* u *is continuously differentiable on* $[0, \infty)$ *and satisfies the Cauchy problem*

$$\begin{cases} u'(t) = \tilde{A}u(t) + X_0 L(u(t)), & t \geqslant 0,\\ u(0) = \varphi. \end{cases} \tag{23}$$

Moreover, the function x *defined on* $[-h, \infty)$ *by*

$$x(t) = \begin{cases} u(t)(0), & \text{if } \ t \geqslant 0,\\ \varphi(t), & \text{if } \ t < 0, \end{cases}$$

is continuously differentiable on $[-h, \infty)$ *and satisfies the Cauchy problem*

$$\begin{cases} x'(t) = A_0 x(t) + Bx'(t-h) + Cx(t-h) + L(x_t), & t \geqslant 0,\\ x_0 = \varphi. \end{cases}$$

Proof. Let u be the solution of (22) on $[0, \infty)$ such that $u(0) = \varphi$. We deduce from Proposition 31 that there exists a unique function v: $[0, \infty) \to \mathscr{C}_X$ which solves the following integral equation

$$v(t) = S'(t)\, \varphi' + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(v(s))\, ds\right).$$

Let $w\colon [0, \infty) \to \mathscr{C}_X$ be the function defined by

$$w(t) = \varphi + \int_0^t v(s)\, ds, \qquad \text{for} \quad t \in [0, \infty).$$

We will show that $w = u$ on $[0, \infty)$.

Using the expression satisfied by v, we obtain

$$w(t) = \varphi + \int_0^t S'(s)\, \varphi'\, ds + \int_0^t \tilde{S}(t-s)\, X_0 L(v(s))\, ds,$$

$$= \varphi + S(t)\, \varphi' + \int_0^t \tilde{S}(t-s)\, X_0 L(v(s))\, ds.$$

On the other hand, we have $\varphi \in D(\tilde{A})$ and $\varphi'(0) = A_0 \varphi(0) + B\varphi'(-h) + C\varphi(-h) + L(\varphi)$, then $\varphi' = \tilde{A}\varphi + X_0 L(\varphi)$. This implies that

$$S(t)\varphi' = \tilde{S}(t)\varphi' = \tilde{S}(t)\, \tilde{A}\varphi + \tilde{S}(t)\, X_0 L(\varphi).$$

Using Corollary 5, we deduce that

$$S(t)\varphi' = S'(t)\, \varphi - \varphi + \tilde{S}(t)\, X_0 L(\varphi).$$

Furthermore, we have

$$\frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(w(s))\, ds\right) = \frac{d}{dt}\left(\int_0^t \tilde{S}(s)\, X_0 L(w(t-s))\, ds\right),$$

$$= \int_0^t \tilde{S}(t-s)\, X_0 L(v(s))\, ds + \tilde{S}(t)\, X_0 L(\varphi).$$

Then

$$w(t) = S'(t)\, \varphi + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, X_0 L(w(s))\, ds\right).$$

We conclude that $w = u$ on $[0, \infty)$. This implies that u is continuously differentiable on $[0, \infty)$.

Consider now the function $g\colon [0, \infty) \to \tilde{\mathscr{C}}_X$ defined by $g(t) = X_0 L(u(t))$ and consider the Cauchy problem

$$\begin{cases} v'(t) = \tilde{A}v(t) + g(t), & t \geqslant 0, \\ v(0) = \varphi. \end{cases} \tag{24}$$

The assumptions imply that $\varphi \in D(\tilde{A})$, $\tilde{A}\varphi + g(0) \in \overline{D(\tilde{A})}$ and g is continuously differentiable on $[0, \infty)$. Using Theorem 10, we deduce that there exists a unique solution on $[0, \infty)$ of Eq. (24). By Theorem 12, we know that this solution is given by

$$y(t) = S'(t)\varphi + \frac{d}{dt}\left(\int_0^t \tilde{S}(t-s)\, g(s)\, ds\right).$$

Proposition 31 implies that $y = u$ on $[0, \infty)$.

If we consider the function x defined on $[-h, \infty)$ by

$$x(t) = \begin{cases} u(t)(0), & \text{if} \quad t \geqslant 0, \\ \varphi(t), & \text{if} \quad t < 0. \end{cases}$$

By virtue of Corollary 32, we have $\int_0^t x(s)\, ds \in D(A_0)$ and

$$x(t) - Bx(t-h) = \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t x(s)\, ds\right) + C\left(\int_0^t x(s-h)\, ds\right) + \int_0^t L(x_s)\, ds, \qquad \text{for} \quad t \in [0, \infty).$$

We have also the existence of

$$\lim_{\sigma \to 0} A_0\left(\frac{1}{\sigma}\int_t^{t+\sigma} u(s)\, ds\right) = x'(t) - Bx'(t-h) - Cx(t-h) - L(x_t),$$

furthermore, the operator A_0 is closed. Then, we obtain $x(t) \in D(A_0)$ and

$$x'(t) - Bx'(t-h) = A_0 x(t) + Cx(t-h) + L(x_t), \qquad \text{for} \quad t \in [0, \infty).$$

We end this section with a result of regularity of the integral solution of Eq. (6). Assume that $T > h$ and $A_0 : D(A_0) \subseteq X \to X$ satisfies (with not necessarily dense domain) the condition

$$\begin{cases} \text{there exist } \beta \in \left]\frac{\pi}{2}, \pi\right[\text{ and } M > 0 \text{ such that if} \\ \lambda \in \mathbf{C} - \{0\} \text{ and } |\arg \lambda| < \beta, \text{ then} \\ \|(\lambda I - A_0)^{-1}\|_{\mathscr{L}(X)} \leqslant \dfrac{M}{|\lambda|}. \end{cases} \tag{25}$$

The condition (25) is stronger than (HY).

We have the following result.

THEOREM 34. *Suppose that* A_0 *satisfies* (25) (*non-densely defined*) *on* X. *Then, for given* $\varphi \in \mathscr{C}_X$, *such that* $\varphi(0) \in \overline{D(A_0)}$, *the integral solution* x *of Eq.* (6) *on* $[0, \infty)$ *is continuously differentiable on* (h, ∞) *and satisfies*

$$x(t) \in D(A_0), \qquad x'(t) \in \overline{D(A_0)} \qquad \textit{and}$$

$$x'(t) - Bx'(t-h) = A_0 x(t) + Cx(t-h) + L(x_t), \qquad \textit{for} \quad t > h.$$

Proof. We know, from [29, p. 487] that A_0 is the generator of an analytic semigroup (not necessarily C_0-semigroup) defined by

$$e^{A_0 t} = \frac{1}{2\pi i} \int_{+C} e^{\lambda t} (\lambda I - A_0)^{-1} \, d\lambda, \qquad t > 0$$

where $+C$ is a suitably oriented path in the complex plan.

Let x be the integral solution on $[0, \infty)$ of Eq. (6), which exists by virtue of Proposition 16, and consider the function $g\colon [0, \infty) \to X$ defined by $g(t) = Cx(t-h) + L(x_t)$. We deduce from [36, p. 106] that

$$x(t) - Bx(t-h) = e^{A_0 t}(\varphi(0) - B\varphi(-h)) + \int_0^t e^{A_0(t-s)} g(s) \, ds,$$

$$\text{for} \quad t \in [0, \infty).$$

By virtue of [37, Theorems 4.4 and 4.5], we deduce that x is continuously differentiable on (h, ∞) and satisfies

$$x(t) \in D(A_0), \qquad x'(t) \in \overline{D(A_0)} \qquad \text{and}$$

$$x'(t) - Bx'(t-h) = A_0 x(t) + Cx(t-h) + L(x_t), \qquad \text{for} \quad t > h.$$

4. RESULTS FOR EQUATION (5)

As we noticed in the introduction, using the same proofs as in Section 3, it is now easy to prove the same results for Eq. (5). The assumption (7) is replaced by some assumptions on the initial conditions. Here we give the main results. Proofs can be nearly duplicated from the ones made in the previous section. So, we leave them to the reader.

THEOREM 35. *If $\varphi \in \mathscr{C}_X$ and $\varphi(0) - B\varphi(-h) \in \overline{D(A_0)}$, then the equation*

$$x(t) = \begin{cases} Bx(t-h) + \dfrac{d}{dt}\left(\displaystyle\int_0^t S_0(t-s)\, Dx(s-h)\, ds\right) & \\ \qquad + S_0'(t)(\varphi(0) - B\varphi(-h)), & \text{if } t \geqslant 0, \\ \varphi(t), & \text{if } t \in [-h, 0], \end{cases} \tag{26}$$

has a unique solution x which is a continuous mapping from $[-h, +\infty) \to X$.

Consider the linear operator Q: $D(Q) \subseteq \mathscr{C}_X \to \mathscr{C}_X$ defined by

$$\begin{cases} D(Q) = \{\varphi \in \mathscr{C}^1([-h, 0], X);\ \varphi(0) - B\varphi(-h) \in D(A_0), \\ \qquad \varphi'(0) = A_0(\varphi(0) - B\varphi(-h)) + B\varphi'(-h) + D\varphi(-h)\}, \\ A\varphi = \varphi'. \end{cases} \tag{27}$$

Under more smoothing properties of φ, we obtain the solution of the following equation

$$\frac{d}{dt}(x(t) - Bx(t-h)) = A_0(x(t) - Bx(t-h)) + Dx(t-h), \qquad t \geqslant 0. \tag{28}$$

THEOREM 36. *Assume that $\varphi \in D(Q)$ and $\varphi'(0) - B\varphi'(-h) \in \overline{D(A_0)}$. Let $x := x(., \varphi)$ be the solution of the integral equation* (26). *Then x is continuously differentiable for all $t \geqslant 0$ and satisfies Eq.* (28).

THEOREM 37. *The operator Q satisfies the condition (HY) on $\mathscr{C}_X$.*

We proceed in a manner similar to the one used for Eq. (6) to define a fundamental integral solution and a variation-of-constants formula associated with Eq. (5).

PROPOSITION 38. *Assume that $\varphi \in \mathscr{C}_X$ and $\varphi(0) - B\varphi(-h) \in \overline{D(A_0)}$. Then, Eq.* (5) *has a unique integral solution x; that is,*

(i) $x \in \mathscr{C}([0, +\infty); X)$,

(ii) $\int_0^t x(s) - Bx(s-h)\, ds \in D(A_0)$, *for* $t \in [0, +\infty)$,

(iii) $x(t) - Bx(t-h)$

$$= \varphi(0) - B\varphi(-h) + A_0\left(\int_0^t x(s) - Bx(s-h)\, ds\right)$$

$$+ D\left(\int_0^t x(s-h)\, ds\right) + L\left(\int_0^t x_s\, ds\right), \qquad \textit{for} \quad t \in [0, +\infty).$$

Furthermore, there exist $\gamma \geqslant 0$ *and* $\mu \in \mathbb{R}$ *such that*

$$\|x_t\| \leqslant \gamma e^{\mu t} \|\varphi\|, \qquad \textit{for} \quad t \geqslant 0.$$

THEOREM 39. *Given* $\varphi \in \mathscr{C}_X$ *such that*

$$\varphi(0) - B\varphi(-h) \in D(A_0), \qquad \varphi' \in \mathscr{C}_X, \qquad \varphi'(0) - B\varphi'(-h) \in \overline{D(A_0)}, \quad \textit{and}$$

$$\varphi'(0) - B\varphi'(-h) = A_0(\varphi(0) - B\varphi(-h)) + D\varphi(-h) + L(\varphi).$$

Then, the integral solution x *of Eq.* (5) *is continuously differentiable on* $[0, +\infty)$ *and satisfies the Cauchy problem*

$$\begin{cases} x'(t) - Bx'(t-h) = A_0(x(t) - Bx(t-h)) + Dx(t-h) + L(x_t), & t \geqslant 0, \\ x_0 = \varphi. \end{cases}$$

5. EXAMPLES

Consider first some notations about spaces of functions with values in a Banach space $(E, \|\cdot\|)$:

- $$\mathscr{C}^\alpha([0, l]; E) = \Bigg\{ u\colon [0, l] \to E;\ [u]_{\mathscr{C}^\alpha([0, l];\, E)} = \sup_{0 \leqslant t \leqslant s \leqslant l} \frac{\|u(t) - u(s)\|}{|t-s|^\alpha} < \infty \Bigg\},$$

  with $\|u\|_{\mathscr{C}^\alpha([0, l];\, E)} = \|u\|_{\mathscr{C}([0, l];\, E)} + [u]_{\mathscr{C}^\alpha([0, l];\, E)}$, $\quad (0 < \alpha < 1)$,

- $$h^\alpha([0, l]; E) = \Bigg\{ u\colon [0, l] \to E;\ \lim_{\delta \to 0} \sup_{0 < |t-s| \leqslant \delta} \frac{\|u(t) - u(s)\|}{|t-s|^\alpha} = 0 \Bigg\},$$

  with $\|u\|_{h^\alpha([0, l];\, E)} = \|u\|_{\mathscr{C}^\alpha([0, l];\, E)}$, $\quad 0 < \alpha < 1$,

- $\mathscr{C}^{\alpha+1}([0, l]; E) = \{u\colon [0, l] \to E;\ u' \in \mathscr{C}^\alpha([0, l]; E)\}$, $\quad 0 < \alpha < 1$.

We consider some examples of operator A_0 with nondense domain verifying the Hille–Yosida condition (see [14]).

(1) Let

$$\begin{cases} X = \mathscr{C}([0, l], \mathbb{R}), \\ D(A_0) = \{u \in \mathscr{C}^1([0, l], \mathbb{R});\ u(0) = 0\}, \\ A_0 u = -u'. \end{cases}$$

We have

$$\overline{D(A_0)} = \{u \in \mathscr{C}([0, l], \mathbb{R}); u(0) = 0\} \neq X$$

and A_0 satisfies (HY) on X (with $M_0 = 1$ and $\omega_0 = 0$).

Let $(S_0(t))_{t \geqslant 0}$ be the integrated semigroup on X generated by A_0. In view of the definition of $(S_0(t))_{t \geqslant 0}$, we have

$$((\lambda I - A_0)^{-1} y)(x) = \lambda \int_0^{+\infty} e^{-\lambda t}(S_0(t)\, y)(x)\, dt.$$

On the other hand, solving the equation

$$(\lambda I - A_0) z = y, \qquad \text{where} \quad \lambda > 0, \quad z \in D(A_0) \quad \text{and} \quad y \in X,$$

we obtain

$$((\lambda I - A_0)^{-1} y)(x) = z(x) = \int_0^a e^{-\lambda t} y(x - t)\, dt.$$

Integrating by parts one obtains

$$((\lambda I - A_0)^{-1} y)(x) = e^{-\lambda x} \int_0^x y(t)\, dt + \int_0^x e^{-\lambda t} \left(\int_{x-t}^x y(s)\, ds \right) dt.$$

By uniqueness of Laplace transform, we obtain

$$(S_0(t)\, y)(a) = \begin{cases} \displaystyle\int_0^x y(s)\, ds, & \text{if} \quad x \leqslant t, \\ \displaystyle\int_{x-t}^x y(s)\, ds, & \text{if} \quad x \geqslant t. \end{cases}$$

(2) Let

$$\begin{cases} X = \{u \in \mathscr{C}^\alpha([0, l], \mathbb{R}); u(0) = 0\}, \qquad 0 < \alpha < 1, \\ D(A_0) = \{u \in \mathscr{C}^{1+\alpha}([0, l], \mathbb{R}); u(0) = u'(0) = 0\}, \\ A_0 u = -u'. \end{cases}$$

We have

$$\overline{D(A_0)} = \{u \in h^\alpha([0, l], \mathbb{R}); u(0) = 0\} \neq X$$

and A_0 satisfies (HY) on X (with $M_0 = 1$ and $\omega_0 = 0$).

(3) Let

$$\begin{cases} X = \mathscr{C}([0, l], \mathbb{R}), \\ D(A_0) = \{u \in \mathscr{C}^2([0, l], \mathbb{R});\ u(0) = u(l) = 0\}, \\ A_0 u = u''. \end{cases}$$

We have

$$\overline{D(A_0)} = \{u \in \mathscr{C}([0, l], \mathbb{R});\ u(0) = u(l) = 0\} \neq X$$

and A_0 satisfies (HY) on X.

(4) Let Ω be a bounded open set of $\mathbb{R}^n$ with regular boundary Γ and define

$$\begin{cases} X = \mathscr{C}(\bar{\Omega}, \mathbb{R}), \\ D(A_0) = \{u \in \mathscr{C}(\bar{\Omega}, \mathbb{R});\ u = 0 \text{ on } \Gamma;\ \Delta u \in \mathscr{C}(\bar{\Omega}, \mathbb{R})\}, \\ A_0 u = \Delta u, \end{cases}$$

here Δ is the Laplacian in the sense of distributions on Ω.

We have

$$\overline{D(A_0)} = \{u \in \mathscr{C}(\bar{\Omega}, \mathbb{R});\ u = 0 \text{ on } \Gamma\} \neq X$$

and A_0 satisfies (HY) on X.

(5) Let

$$X = W_0^{1,p}(0, l; \mathscr{C}(\bar{\Omega}, \mathbb{R})),$$

$$D(A_0) = \{u \in \mathscr{C}([0, l], D(\Delta)) \cap \mathscr{C}^1([0, l], \mathscr{C}(\bar{\Omega}, \mathbb{R}));\ u(0) = 0 \quad \text{and}$$

$$\Delta u - u' \in W_0^{1,p}(0, l; \mathscr{C}(\bar{\Omega}, \mathbb{R}))\},$$

and

$$A_0 u = \Delta u - u'.$$

We have $\overline{D(A_0)} \neq X$ and A_0 satisfies (HY) on X.

(6) Let

$$\begin{cases} X = \{u \in \mathscr{C}^\alpha([0, l], \mathscr{C}(\bar{\Omega}, \mathbb{R}));\ u(0) = 0\}, \qquad 0 < \alpha < 1, \\ D(A_0) = \{u \in \mathscr{C}^\alpha([0, l], D(\Delta)) \cap \mathscr{C}^{\alpha+1}([0, l], \mathscr{C}(\bar{\Omega}, \mathbb{R}));\ u(0) = u'(0) = 0\}, \\ A_0 u = \Delta u - u'. \end{cases}$$

We have $\overline{D(A_0)} \neq X$ and A_0 satisfies (HY) on X.

One can also consider the periodic versions of examples (5) and (6).

(7) Let

$$\begin{cases} X = L^{\infty}(\mathbb{R}), \\ D(A_0) = \{u \in L^{\infty}(\mathbb{R}),\ u \text{ is absolutely continuous and } u' \in L^{\infty}(\mathbb{R})\}, \\ A_0 u = -u'. \end{cases}$$

We have $\overline{D(A_0)} \neq X$ and A_0 satisfies (HY) on X.

Finally, we give two examples of linear partial neutral functional differential-difference equations with nondense domain.

(8) Consider the following system

$$\begin{cases} \dfrac{\partial}{\partial t}\left(u(t,x) - \displaystyle\int_0^x k(x-s)\, u(t-h,s)\, ds\right) \\ \qquad = -\dfrac{\partial}{\partial x} u(t,x) + F(u_t(.,x)), & t \geqslant 0, & x \in [0,l], \\ u(t,0) = 0, & t \geqslant 0, \\ u(\theta,x) = \varphi(\theta,x), & \theta \in [-h,0], & x \in [0,l]. \end{cases} \tag{29}$$

where $l > 0$, $\varphi \in \mathscr{C}_X := \mathscr{C}([-h,0],X)$, $X = \mathscr{C}([0,l],\mathbb{R})$, $k \in \mathscr{C}^1([0,l],\mathbb{R})$ and F is a continuous linear functional from $\mathscr{C}([-h,0],\mathbb{R})$ into $\mathbb{R}$.

By setting $U(t) = u(t,\cdot)$ and $BU(t)(x) = \int_0^x k(x-s)\, u(t,s)\, ds$, Eq. (29) reads

$$\begin{cases} \dfrac{d}{dt}(U(t) - BU(t-h)) = A_0 U(t) + L(U_t), & t \geqslant 0, \\ U(0) = \varphi, \end{cases}$$

where $A_0\colon D(A_0) \subseteq X \to X$ is the linear operator given in the example (1) and $L\colon \mathscr{C}_X \to X$ is the function defined by

$$L(\varphi)(x) = F(\varphi(\cdot,x)), \qquad \text{for} \quad t \geqslant 0, \qquad \varphi \in \mathscr{C}_X \qquad \text{and} \qquad x \in [0,l].$$

In this case, we have

$$\operatorname{Range}(B) \subseteq D(A_0).$$

THEOREM 40. *For a given $\varphi \in \mathscr{C}_X$, such that*

$$\varphi(0,0) = 0,$$

Problem (29) *has a unique integral solution, i.e.*,

(i) $u \in \mathscr{C}([0, +\infty); X)$, $\int_0^t u(s, \cdot)\, ds \in \mathscr{C}^1([0, l], \mathbb{R})$, and $\int_0^t u(s, 0)\, ds = 0$, for $t \geqslant 0$,

(ii)
$$\begin{cases} u(t, x) = \displaystyle\int_0^x k(x-s)\, u(t-h, s)\, ds + \varphi(0, x) \\ \quad - \displaystyle\int_0^x k(x-s)\, \varphi(-h, s)\, ds \\ \quad - \dfrac{\partial}{\partial x}\left(\displaystyle\int_0^t u(s, x)\, ds\right) + F\left(\displaystyle\int_0^t u_s(., x)\, ds\right), \\ \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad x \in [0, l], \\ u(t, x) = \varphi(t, x), \qquad \text{for} \quad t \in [-h, 0] \quad \text{and} \quad x \in [0, l]. \end{cases} \tag{30}$$

Proof. The assumptions of Theorem 40 imply that $\varphi \in \mathscr{C}_X$ and $\varphi(0, \cdot) \in \overline{D(A_0)}$. Consequently, from Corollary 32, we deduce that there exists a unique function v: $[0, +\infty) \to \mathscr{C}_X$ which solves the integral equation (30). ∎

THEOREM 41. *For given $\varphi \in \mathscr{C}_X$ such that*

$$\frac{\partial \varphi}{\partial t} \in \mathscr{C}_X, \qquad \varphi(0, \cdot) \in \mathscr{C}^1([0, l], \mathbb{R}),$$

$$\varphi(0, 0) = \frac{\partial \varphi}{\partial t}(0, 0) = 0 \qquad \textit{and}$$

$$\frac{\partial}{\partial t}\varphi(0, x) = \int_0^x k(x-s)\, \frac{\partial}{\partial t}\varphi(-h, s)\, ds - \frac{\partial}{\partial x}\varphi(0, x) + F(\varphi(., x)), \qquad \textit{for} \quad x \in [0, l].$$

the solution u of Eq. (30) *is continuously differentiable on* $[0, +\infty) \times [0, l]$ *and is equal to the unique solution of Problem* (29).

Proof. The assumptions of Theorem 41 imply that $\varphi \in D(A)$ and $(\partial\varphi/\partial t)(0, .) \in \overline{D(A_0)}$.

The proof follows from Theorem 33. ∎

Remark 2. We can also consider the example 8 in the form of Eq. (5) with a general operator B and with additional assumptions on φ.

(9) Let us consider now the following PNFDDE

$$\begin{cases} \dfrac{\partial}{\partial t}(u(t,.)-Bu(t-h,.)) = \Delta(u(t,.)-Bu(t-h,.))+F(u_t), & t \geqslant 0, \\ u(t,x)=0, \qquad t\geqslant 0, & x\in\partial\Omega, \\ u(\theta,x)=\varphi(\theta,x), \qquad \theta\in[-h,0], & x\in\Omega, \end{cases} \tag{31}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set with regular boundary $\partial\Omega$, Δ is the Laplace operator in the sense of distributions on Ω, φ is a given function on $\mathscr{C}_X := \mathscr{C}([-h,0],X)$, with $X=\mathscr{C}(\bar{\Omega},\mathbb{R})$ and $B\in\mathscr{L}(X)$.

Problem (31) can be reformulated as an abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}(U(t)-BU(t-h)) = A_0(U(t)-BU(t-h))+L(U_t), & t\geqslant 0, \\ U(0)=\varphi, \end{cases}$$

where $A_0\colon D(A_0)\subseteq X\to X$ is the linear operator given in example (4) and $L\colon\mathscr{C}_X\to X$ is the function defined by

$$L(\varphi)(x)=F(\varphi(\cdot,x)), \qquad \text{for} \quad t\geqslant 0, \quad \varphi\in\mathscr{C}_X \quad \text{and} \quad x\in\bar{\Omega}.$$

Using the results of Section 4, we obtain the following theorems (all the assumptions are satisfied).

THEOREM 42. *For a given $\varphi\in\mathscr{C}_X$, such that*

$$\varphi(0,\cdot)-B\varphi(-h,\cdot)=0, \qquad \text{on } \partial\Omega,$$

there exists a unique integral solution $u\colon[0,+\infty)\to X$ of the partial differential equation (31), *i.e.*,

(i) $u(t,x)=\varphi(t,x)$ *if* $t\in[-h,0]$, $x\in\bar{\Omega}$,

(ii) $u\in\mathscr{C}([0,+\infty);X)$, $\Delta(\int_0^t (u(s,\cdot)-Bu(s-h,\cdot))\,ds)\in\mathscr{C}(\bar{\Omega},\mathbb{R})$ *and* $\int_0^t (u(s,\cdot)-Bu(s-h,\cdot))\,ds=0$, *on* $\partial\Omega$, *for* $t\geqslant 0$

(iii) $u(t,x)-Bu(t-h,x)$

$$= \varphi(0,x)-B\varphi(-h,x)+\Delta\left(\int_0^t (u(s,\cdot)-Bu(s-h,\cdot))\,ds\right) + \int_0^t F(u_s(\cdot,x))\,ds, \qquad \text{for} \quad t\geqslant 0, \quad x\in\bar{\Omega}.$$

Theorem 43. *For a given $\varphi \in \mathscr{C}_X$ such that*

$$\frac{\partial \varphi}{\partial t} \in \mathscr{C}_X, \qquad \Delta(\varphi(0,\cdot) - B\varphi(-h,\cdot)) \in \mathscr{C}(\bar{\Omega}, \mathbb{R}),$$

$$\varphi(0,\cdot) - B\varphi(-h,\cdot) = \frac{\partial}{\partial t}(\varphi(0,\cdot) - B\varphi(-h,\cdot)) = 0 \qquad \text{on } \partial\Omega \qquad \text{and}$$

$$\frac{\partial}{\partial t}(\varphi(0,x) - B\varphi(-h,x)) = \Delta(\varphi(0,x) - B\varphi(-h,x)) + F(\varphi(\cdot,x)),$$

$$\text{for} \quad x \in \bar{\Omega}.$$

There is a unique function u defined on $[-h, +\infty) \times \bar{\Omega}$, such that $x = \varphi$ on $[-h, 0] \times \bar{\Omega}$ and satisfies Eq. (31) *on* $[0, +\infty) \times \bar{\Omega}$.

REFERENCES

1. M. Adimy, Bifurcation de Hopf locale par les semi-groupes intégrés, *C.R. Acad. Sci. Paris* **311**(I) (1990), pp. 423–428.
2. M. Adimy, Integrated semigroups and delay differential equations, *J. Math. Anal. Appl.* **177** (1993), 125–134.
3. M. Adimy, Abstract semilinear functional differential equations with non-dense domain, preprint URA 1204 Pau 95/18.
4. M. Adimy and O. Arino, Bifurcation de Hopf globale pour des équations à retard par des semi-groupes intégrés, *C.R. Acad. Sci. Paris* **317**(I) (1993), pp. 767–772.
5. M. Adimy and K. Ezzinbi, Equations de type neutre et semi-groupes intégrés, *C.R. Acad. Sci. Paris* **318**(I) (1994), 529–534.
6. M. Adimy and K. Ezzinbi, Semi-groupes intégrés et équations différentielles à retard en dimension infinie, *C.R. Acad. Sci. Paris* **323**(I) (1996), 481–486.
7. W. Arendt, Resolvent positive operators and integrated semigroup, *Proc. London Math. Soc. 3* **54** (1987), 321–349.
8. W. Arendt, Vector valued Laplace transforms and Cauchy problems, *Israel J. Math.* **59** (1987), 327–352.
9. O. Arino and E. Sanchez, Linear theory of abstract functional differential equations of retarded type, *J. Math. Anal. Appl.* **191** (1995), 547–571.
10. R. Bellman and K. Cooke, "Differential Difference Equations," Academic Press, San Diego, 1963.
11. S. Busenberg and B. Wu, Convergence theorems for integrated semigroups, *Differential Integral Equations* **5**(3) (1992), 509–520.
12. K. L. Cooke and D. W. Krumme, Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations, *J. Math. Anal. Appl.* **24** (1968), 372–387.
13. M. A. Cruz and J. K. Hale, Asymptotic behavior of neutral functional differential equations, *Arch. Rational Mech. Anal.* **34** (1969), 331–353.
14. G. Da Prato and E. Sinestrari, Differential operators with non-dense domains, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **14** (1987), 285–344.
15. R. Datko, Linear autonomous neutral differential equations in a Banach space, *J. Differential Equations* **25** (1977), 258–274.
16. K. Ezzinbi and H. Tamou, Abstract functional differential equations, preprint.

17. W. E. Fitzgibbon, Semilinear functional differential equations in Banach space, *J. Differential Equations* **29** (1978), 1–14.
18. A. Grabosch and U. Moustakas, A semigroup approach to retarded differential equations, *in* "One-Parameter Semigroups of Positive Operators" (R. Nagel, Ed.), Lecture Notes in Math., Vol. 1184, pp. 219–232, Springer-Verlag, Berlin/New York, 1986.
19. J. K. Hale, A class of neutral equations with the fixed-point property, *Proc. Nat. Acad. Sci. U.S.A.* **67** (1970), 136–137.
20. J. K. Hale, Critical cases for neutral functional differential equations, *J. Differential Equations* **10** (1971), 59–82.
21. J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, Berlin/New York, 1977.
22. J. K. Hale, Partial neutral functional differential equations, *Rev. Roumaine Math. Pures Appl.* **39** (1994), 339–344.
23. J. K. Hale, Coupled oscillators on a circle, *Resenhas* **1**(4) (1994), 441–457.
24. J. K. Hale, X. B. Lin, and G. Raugel, Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, *Math. Comp.* **50** (1988), 89–123.
25. J. K. Hale and S. Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, Berlin/New York, 1993.
26. J. K. Hale and K. R. Meyer, A class of functional equations of neutral type, *Mem. Amer. Math. Soc.* **76** (1967).
27. D. Henry, Linear autonomous neutral functional differential equations, *J. Differential Equations* **15** (1974), 106–128.
28. M. Hieber, "Integrated Semigroups and Differential Operators on L^p," Dissertation, 1989.
29. T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, Berlin, 1966.
30. K. Kunisch and W. Schappacher, Variation of constants formula for partial differential equations with delay, *Nonlinear Anal.* **5** (1981), 123–142.
31. K. Kunisch and W. Schappacher, Necessary conditions for partial differential equations with delay to generate a C_0-semigroups, *J. Differential Equations* **50** (1983), 49–79.
32. H. Kellermann, "Integrated Semigroups," Dissertation, Tübingen, 1986.
33. H. Kellermann and M. Hieber, Integrated semigroup, *J. Funct. Anal.* **15** (1989), 160–180.
34. M. Memory, Invariant manifolds for partial functional differential equations, *in* "Mathematical Population Dynamics" (O. Arino, D. E. Axelrod, and M. Kimmel, Eds.), pp. 223–232, Dekker, 1991.
35. F. Neubrander, Integrated semigroups and their applications to the abstract Cauchy problems, *Pacific J. Math.* **135** (1988), 111–155.
36. A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, Berlin/New York, 1983.
37. E. Sinestrari, On the abstract Cauchy problem of parabolic type in spaces of continuous functions, *J. Math. Anal. Appl.* **107** (1985), 16–66.
38. H. Thieme, Integrated semigroups and integrated solutions to abstract Cauchy problems, *J. Math. Anal. Appl.* **152** (1990), 416–447.
39. C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974), 395–418.
40. G. F. Webb, Asymptotic stability for abstract functional differential equations, *Proc. Amer. Math. Soc.* **54** (1976), 225–230.
41. J. Wu, "Theory and Applications of Partial Functional Differential Equations," Springer-Verlag, Berlin/New York, 1996.
42. J. Wu and H. Xia, Self-sustained oscillations in a ring array of coupled lossless transmission lines, *J. Differential Equations* **124** (1996), 247–278.
43. J. Wu and H. Xia, Rotating waves in neutral partial functional differential equations, preprint.